Physics of High Intensity Laser Plasma Interactions

Varenna Summer School on Laser-Plasma Acceleration

20–25 June 2011 | Paul Gibbon
Course outline

- Lecture 1: Introduction – Definitions and Thresholds
- Lecture 2: Interaction with Underdense Plasmas
- Lecture 3: Interaction with Solids
- Lecture 4: Numerical Simulation of Laser-Plasma Interactions
- Lectures 5 & 6: Tutorial on Particle-in-Cell Simulation
Physics of High Intensity Laser Plasma Interactions
Part I: Definitions and thresholds

20–25 June 2011  |  Paul Gibbon
Lecture 1: Definitions and thresholds

Introduction

Field ionization

Relativistic threshold

Plasma Debye length

Plasma frequency

Further reading
Figure 1: Progress in peak intensity since the invention of the laser.
Extreme conditions: nonlinear, strong-field science

- Ordinary matter — solid, liquid or gas — rapidly ionized when subjected to high intensity irradiation.
- Electrons released are then immediately caught in the laser field.
- Oscillate with a characteristic energy which dictates the subsequent interaction physics.
- Basis for laser-based particle accelerator schemes and short-wavelength radiation sources.
Electrons in intense electromagnetic fields

prehistory

- Volkov (1935): electron ‘dressed’ by field – relativistic mass increase

\[ q = eE_L m \omega c \]  

\( e \) = electron charge, \( m \) = electron mass, \( c \) = speed of light; \( E_L \) = laser electric field strength; \( \omega \) = light frequency.
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- Schwinger (1949): cyclotron radiation

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Ostriker & Gunn (1969) – electron dynamics in vicinity of pulsars: \( q \sim 100 \)
Electrons in intense electromagnetic fields
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- Schwinger (1949): cyclotron radiation
- Invention of laser (1960): theoretical works on electron dynamics

Figure of merit $q$:

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Where $e$ = electron charge, $m$ = electron mass, $c$ = speed of light; $E_L$ = laser electric field strength; $\omega$ = light frequency.
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Field ionization

At the Bohr radius

\[ a_B = \frac{\hbar^2}{me^2} = 5.3 \times 10^{-9} \text{ cm}, \]

the electric field strength is:

\[ E_a = \frac{e}{4\pi\varepsilon_0 a_B^2} \]

\[ \simeq 5.1 \times 10^9 \text{ Vm}^{-1}. \] (2)
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This leads to the atomic intensity:

\[ I_a = \frac{\varepsilon_0 c E_a^2}{2} \]

\[ \simeq 3.51 \times 10^{16} \text{ Wcm}^{-2}. \] (3)
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A laser intensity of \( I_L > I_a \) will guarantee ionization for any target material, though in fact this can occur well below this threshold value via multiphoton effects.
Tunneling ionization

- Keldysh (1965) and Perelomov (1966): introduced a parameter $\gamma$ separating the multiphoton and tunneling regimes, given by:

$$
\gamma = \omega_L \sqrt{\frac{2E_{\text{ion}}}{I_L}} \sim \sqrt{\frac{E_{\text{ion}}}{\Phi_{\text{pond}}}}.
$$

(4)

where

$$
\Phi_{\text{pond}} = \frac{e^2 E_L^2}{4m\omega_L^2}
$$

(5)

is the *ponderomotive potential* of the laser field.
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Regimes

$\gamma < 1 \Rightarrow$ tunneling – strong fields, long wavelengths
$\gamma > 1 \Rightarrow$ MPI – short wavelengths
Figure 2: a) Schematic picture of tunneling or barrier-suppression ionization by a strong external electric field.
Barrier suppression model II

- Coulomb potential modified by a stationary, homogeneous electric field, see Fig. 13:

\[ V(x) = -\frac{Ze^2}{x} - e\varepsilon x. \]

⇒ suppressed on RHS of the atom, and for \( x \gg x_{\text{max}} \) is lower than the binding energy of the electron.
Barrier suppression model II

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⇒ suppressed on RHS of the atom, and for \( x \gg x_{\text{max}} \) is lower than the binding energy of the electron.

- If the barrier falls below \( E_{\text{ion}} \), the electron will escape spontaneously
  ⇒ \textit{barrier suppression} (BS) ionization.

- Set \( dV(x)/dx = 0 \) to determine the position of the barrier:

\[ x_{\text{max}} = (Ze/\varepsilon), \]
Barrier suppression model III

- Set \( V(x_{\text{max}}) = E_{\text{ion}} \) to get the threshold field strength for BS:
  \[
  \varepsilon_c = \frac{E_{\text{ion}}^2}{4Ze^3}.
  \]  
  (6)

- Equate critical field to the peak electric field of the laser – appearance intensity for ions created with charge \( Z \):
  \[
  I_{\text{app}} = \frac{c}{8\pi} \varepsilon_c^2 = \frac{cE_{\text{ion}}^4}{128\pi Z^2 e^6},
  \]  
  (7)
Barrier suppression model III

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- Equate critical field to the peak electric field of the laser – *appearance intensity* for ions created with charge $Z$:
  \[ I_{\text{app}} = \frac{c}{8\pi} \varepsilon_c^2 = \frac{cE_{\text{ion}}^4}{128\pi Z^2 e^6}, \] (7)

Appearance intensity

\[ I_{\text{app}} \simeq 4 \times 10^9 \left( \frac{E_{\text{ion}}}{\text{eV}} \right)^4 Z^{-2} \text{ Wcm}^{-2} \] (8)

- $E_{\text{ion}}$ is the ionization potential of the ion or atom with charge $(Z - 1)$. 
# Appearance intensities of selected ions

<table>
<thead>
<tr>
<th>Ion</th>
<th>$E_{ion}$ (eV)</th>
<th>$I_{app}$ (Wcm$^{-2}$)</th>
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<tr>
<td>H$^+$</td>
<td>13.61</td>
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**Table 1:** BS ionization model – Eq. (8).
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Table 1: BS ionization model – Eq. (8).

Figure 3: Auguste et al., J. Phys. B (1992)
Relativistic field strengths

Classical equation of motion for an electron exposed to a linearly polarized laser field $E = \hat{y}E_0 \sin \omega t$:

$$\frac{dv}{dt} \simeq -\frac{eE_0}{m_e} \sin \omega t$$

Dimensionless oscillation amplitude, or 'quiver' velocity:

$$a_0 \equiv v_0 \equiv p_0 m_e c \equiv \frac{eE_0 m_e}{\gamma c}$$
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Definitions and thresholds
Relativistic threshold
Relativistic field strengths

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Dimensionless oscillation amplitude, or ’quiver’ velocity:

\[
a_0 \equiv \frac{v_{os}}{c} \equiv \frac{p_{os}}{m_e c} \equiv \frac{eE_0}{m_e \omega c}
\]  \hspace{1cm} (10)
Relativistic intensity

The laser intensity $I_L$ and wavelength $\lambda_L$ are related to $E_0$ and $\omega$ by:

$$I_L = \frac{1}{2} \varepsilon_0 c E_0^2; \quad \lambda_L = \frac{2\pi c}{\omega}$$

Substituting these into (10) we find (Exercise):

$$a_0 \approx 0.85 \left( I_{18} \lambda_{\mu m}^2 \right)^{-1/2}, \quad (11)$$

where $I_{18} = I_L \times 10^{18}$ W cm$^{-2}$; $\lambda_{\mu m} = \lambda_L \mu m$.

Implies that for $I_L \geq 10^{18}$ W cm$^{-2}$, $\lambda_L \approx 1 \mu m$, we will have relativistic electron velocities, or $a_0 \approx 1$.
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Definitions and thresholds
Plasma definitions: the Debye length

A plasma created by field-ionization of a gas or solid will initially be *quasi-neutral*. This means that the number densities of electrons and ions with charge state $Z$ are locally balanced:

$$n_e \sim Zn_i.$$
Plasma definitions: the Debye length

A plasma created by field-ionization of a gas or solid will initially be \textit{quasi-neutral}. This means that the number densities of electrons and ions with charge state $Z$ are locally balanced:

$$n_e \sim Z n_i.$$  

Any local disturbance in the charge distribution will be rapidly neutralised by the lighter, faster electrons – \textit{Debye shielding}: shields potential around exposed charge:  

$$\phi_D = \frac{e^{-r/\lambda_D}}{r}.$$
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Debye length

$$\lambda_D = \left( \frac{\varepsilon_0 k_B T_e}{e^2 n_e} \right)^{1/2} = 743 \left( \frac{T_e}{\text{eV}} \right)^{1/2} \left( \frac{n_e}{\text{cm}^{-3}} \right)^{-1/2} \text{ cm} \quad (12)$$
Plasma classification

An *ideal* plasma has many particles per Debye sphere:

\[ N_D \equiv n_e \frac{4\pi}{3} \lambda_D^3 \gg 1. \]  

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Plasma frequency

For a thermal plasma with temperature $T_e$ one can also define a characteristic \textit{response time} (eg: to disturbances from external laser fields or particle beams):

$$t_D \simeq \frac{\lambda_D}{v_t} = \left( \frac{\varepsilon_0 k_B T_e}{e^2 n_e} \cdot \frac{m}{k_B T_e} \right)^{1/2} = \left( \frac{e^2 n_e}{\varepsilon_0 m_e} \right)^{-1/2}.$$
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Electron plasma frequency

$$\omega_p \equiv \left( \frac{e^2 n_e}{\varepsilon_0 m_e} \right)^{1/2} \simeq 5.6 \times 10^4 \left( \frac{n_e}{\text{cm}^{-3}} \right)^{1/2} \text{s}^{-1}. \quad (14)$$
Underdense and overdense plasmas

If the plasma response time is shorter than the period of a external electromagnetic field (such as a laser), then this radiation will be *shielded out*. 
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**Figure 5:** Underdense, $\omega > \omega_p$: plasma acts as nonlinear refractive medium
Underdense and overdense plasmas

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\textbf{Figure 5:} Underdense, $\omega > \omega_p$: plasma acts as nonlinear refractive medium

\textbf{Figure 6:} Overdense, $\omega < \omega_p$: plasma acts like mirror
The critical density

To make this more quantitative, consider ratio:

\[
\frac{\omega_p^2}{\omega^2} = \frac{e^2 n_e}{\varepsilon_0 m_e} \cdot \frac{\lambda^2}{4\pi^2 c^2}.
\]
The critical density

To make this more quantitative, consider ratio:

$$\frac{\omega_p^2}{\omega^2} = \frac{e^2 n_e}{\varepsilon_0 m_e} \cdot \frac{\lambda^2}{4\pi^2 c^2}.$$ 

Setting this to unity defines the wavelength for which $n_e = n_c$, or the critical density

$$n_c \simeq 10^{21} \lambda^{-2} \text{ cm}^{-3} \quad (15)$$

above which radiation with wavelengths $\lambda > \lambda_\mu$ will be reflected. cf: radio waves in ionosphere.
## Laser-generated plasmas

<table>
<thead>
<tr>
<th>Target material</th>
<th>Electron density $n_e$ (cm$^{-3}$)</th>
<th>$n_e/n_c$ $(800$nm$)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capillary discharge</td>
<td>$10^{16}$</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>Gas jet</td>
<td>$10^{18}$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>Foam/aerogel</td>
<td>$10^{21}$</td>
<td>$0.1 - 5$</td>
</tr>
<tr>
<td>Frozen H</td>
<td>$10^{22}$</td>
<td>36</td>
</tr>
<tr>
<td>CH foil</td>
<td>$5 \times 10^{23}$</td>
<td>600</td>
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</tbody>
</table>

Table 2: Electron densities in typical laser-produced plasmas
Further reading

Summary of Lecture 1

Introduction

Field ionization

Relativistic threshold

Plasma Debye length

Plasma frequency

Further reading
Physics of High Intensity Laser Plasma Interactions
Part II: Interaction with Underdense Plasmas

20–25 June 2011 | Paul Gibbon
Lecture 2: Interaction with Underdense Plasmas

Plasma response
Cold fluid equations
  EM waves
  Plasma waves
EM wave propagation
  Nonlinear refraction
  Self focussing
  SF power threshold
  Ponderomotive channel formation
Plasma wave propagation
  Dispersion
  Numerical solutions
  Wave breaking
Wakefield excitation
  Electron acceleration
  Bench-Top Particle Accelerators
Ionized gases: when is plasma response important?

Simultaneous field ionization of many atoms produces a plasma with electron density $n_e$, temperature $T_e \sim 1 - 10$ eV. Collective effects important if

$$\omega_p \tau_L > 1$$

Example (Gas jet)

$\tau_L = 100$ fs, $n_e = 10^{17}$ cm$^{-3}$ $\rightarrow \omega_p \tau_L = 1.8$.

Typical gas jets: $P \sim 1$ bar; $n_e = 10^{18} - 10^{19}$ cm$^{-3}$.

Recall that from Eq. 15, critical density for glass laser $n_c(1 \mu)$ = 10$^{21}$ cm$^{-3}$. Gas-jet plasmas are therefore underdense, since

$$\omega^2 / \omega_p^2 = n_e / n_c < 1.$$
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Exploit plasma effects for: short-wavelength radiation; nonlinear refractive properties; high electric/magnetic fields.
Wave propagation

The starting point for most analyses of nonlinear wave propagation phenomena is the Lorentz equation of motion for the electrons in a cold ($T_e = 0$), unmagnetized plasma, together with Maxwell’s equations.
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We also make two assumptions:

1. The ions are initially assumed to be singly charged ($Z = 1$) and are treated as a immobile ($v_i = 0$), homogeneous background with $n_0 = Zn_i$. 
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We also make two assumptions:

1. The ions are initially assumed to be singly charged ($Z = 1$) and are treated as a immobile ($v_i = 0$), homogeneous background with $n_0 = Z n_i$.

2. Thermal motion is neglected – justified for underdense plasmas because the temperature remains small compared to the typical oscillation energy in the laser field.
Lorentz-Maxwell equations

Starting equations (SI units) are as follows

\[
\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla)p = -e(E + \mathbf{v} \times \mathbf{B}),
\] (16)

where \(p = \gamma m_e v\) and \(\gamma = \left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2}\). 

Interaction with Underdense Plasmas

Cold fluid equations
Lorentz-Maxwell equations

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\[ \nabla \cdot \mathbf{E} = \frac{e}{\varepsilon_0} (n_0 - n_e), \quad (17) \]
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\nabla \cdot E = \frac{e}{\varepsilon_0}(n_0 - n_e), \quad (17)
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\[
\nabla \times E = -\frac{\partial B}{\partial t}, \quad (18)
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\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{18}
\]

\[
c^2 \nabla \times \mathbf{B} = -\frac{e}{\varepsilon_0} n_e \mathbf{v} + \frac{\partial \mathbf{E}}{\partial t}, \tag{19}
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Starting equations (SI units) are as follows

\[
\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla)p = -e(E + \mathbf{v} \times \mathbf{B}),
\]

(16)

\[
\nabla \cdot E = \frac{e}{\varepsilon_0} (n_0 - n_e),
\]

(17)

\[
\nabla \times E = -\frac{\partial B}{\partial t},
\]

(18)

\[
c^2 \nabla \times \mathbf{B} = -\frac{e}{\varepsilon_0} n_e \mathbf{v} + \frac{\partial \mathbf{E}}{\partial t},
\]

(19)

\[
\nabla \cdot \mathbf{B} = 0,
\]

(20)

where \( p = \gamma m_e \mathbf{v} \) and \( \gamma = \left(1 + \frac{p^2}{m_e c^2}\right)^{1/2} \).
To simplify matters we first assume a plane-wave geometry like that above. A laser pulse can thus be described by the electromagnetic fields $\mathbf{E}_L = (0, E_y, 0); \mathbf{B}_L = (0, 0, B_z)$. 

Exercise: From Eq. (16) one can show that the transverse electron momentum is then simply given by:

$$p_y = eA_y,$$

(21)

where $E_y = \partial A_y/\partial t$. This relation expresses conservation of canonical momentum.
Electromagnetic waves

To simplify matters we first assume a plane-wave geometry like that above. A laser pulse can thus be described by the electromagnetic fields \( E_L = (0, E_y, 0); \ B_L = (0, 0, B_z) \).

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where \( E_y = \partial A_y/\partial t \). This relation expresses conservation of canonical momentum.
The EM wave equation I

Substitute $E = -\nabla \phi - \partial A / \partial t$; $B = \nabla \times A$ into Ampère Eq.(19):

$$c^2 \nabla \times (\nabla \times A) + \frac{\partial^2 A}{\partial t^2} = \frac{J}{\varepsilon_0} - \nabla \frac{\partial \phi}{\partial t},$$

where the current $J = -en_e v$. 
The EM wave equation I

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where the current $J = -en_e v$.

Now we use a bit of vectorial magic, splitting the current into rotational (solenoidal) and irrotational (longitudinal) parts:

$$J = J_\perp + J_\parallel = \nabla \times \Pi + \nabla \Psi$$

from which we can deduce (see Jackson!):

$$J_\parallel - \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} = 0.$$
The EM wave equation II

Now apply Coulomb gauge $\nabla \cdot A = 0$ and $v_y = eA_y / \gamma$ from (21), to finally get:

\[
\frac{\partial^2 A_y}{\partial t^2} - c^2 \nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y. \tag{22}
\]
The EM wave equation II

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\[ \frac{\partial^2 A_y}{\partial t^2} - c^2 \nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y. \quad (22) \]

The nonlinear source term on the RHS contains two important bits of physics:

- $n_e = n_0 + \delta n \rightarrow$ Coupling to plasma waves
- $\gamma = \sqrt{1 + p^2 / m_e c^2} \rightarrow$ Relativistic effects
Electrostatic (Langmuir) waves I

Taking the \textit{longitudinal} (x)-component of the momentum equation (16) gives:

\[ \frac{d}{dt}(\gamma m_e v_x) = -eE_x - \frac{e^2}{2m_e \gamma} \frac{\partial A_y^2}{\partial x} \]
Electrostatic (Langmuir) waves I

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\]

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\[
0 = -\frac{e}{\varepsilon_0} n_e v_x + \frac{\partial E_x}{\partial t},
\]
Electrostatic (Langmuir) waves I

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\]

We can eliminate \(v_x\) using Ampère’s law (19)\(_x\):

\[
0 = -\frac{e}{\varepsilon_0} n_e v_x + \frac{\partial E_x}{\partial t},
\]

while the electron density can be determined via Poisson’s equation (17):

\[
n_e = n_0 - \frac{\varepsilon_0}{e} \frac{\partial E_x}{\partial x}.
\]
Electrostatic (Langmuir) waves II

The above (closed) set of equations can in principle be solved numerically. For the moment, we simplify things by *linearizing* the plasma quantities:

\[
n_e \approx n_0 + n_1 + \ldots
\]

\[
v_x \approx v_1 + v_2 + \ldots
\]

and neglect products like \( n_1 v_1 \) etc. This finally leads to:

**Driven plasma wave**

\[
\left( \frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0} \right) E_x = -\frac{\omega_p^2 e}{2m_e \gamma_0^2} \frac{\partial}{\partial x} A_y^2
\]  

(23)
Electrostatic (Langmuir) waves II

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\[ n_e \simeq n_0 + n_1 + \ldots \]
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\]

(23)

The driving term on the RHS is the relativistic ponderomotive force, with \( \gamma_0 = (1 + a_0^2/2)^{1/2} \).
Cold plasma wave equations: recap

Electromagnetic wave

\[
\frac{\partial^2 A_y}{\partial t^2} - c^2 \nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y
\]

Electrostatic (Langmuir) wave

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Cold plasma wave equations: recap

Electromagnetic wave

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Electrostatic (Langmuir) wave

\[ \left( \frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0} \right) E_x = -\frac{\omega_p^2 e}{2m_e \gamma_0^2} \frac{\partial}{\partial x} A_y^2 \]

These coupled fluid equations describe a vast range of nonlinear laser-plasma interaction phenomena: parametric instabilities, self-focussing, channelling, wakefield excitation, harmonic generation, ...
Dispersion properties: EM waves

This time we switch the plasma oscillations off \((n_e = n_0)\) in Eq.(22) and look for solutions:

\[
A_y = A_0 \sin(\omega t - kx),
\]
Dispersion properties: EM waves

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\]

to obtain

\[
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\]
Dispersion properties: EM waves

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\[
A_y = A_0 \sin(\omega t - kx),
\]

to obtain

\[
\omega^2 = \frac{\omega_p^2}{\gamma_0} + c^2k^2.
\] (24)

From this relation we can derive a

**Nonlinear refractive index**

\[
\eta = \sqrt{\frac{c^2k^2}{\omega^2}} = \left(1 - \frac{\omega_p^2}{\gamma_0\omega^2}\right)^{1/2}
\] (25)
Nonlinear refraction effects

Have so far assumed plane wave approximation for laser pulse – ’photon bullet’
Real laser pulses are created with focusing optics & are subject to:

1. Diffraction due to finite focal spot $\sigma_L$:

$$Z_R = \frac{2\pi \sigma_L^2}{\lambda}$$

2. Ionization effects: refraction due to radial density gradients

3. Relativistic self-focusing. Power threshold:

$$P_c \approx 17 \left( \frac{\omega_0}{\omega_p} \right)^2 \text{GW}, \quad (26)$$

4. Ponderomotive channelling

All nonlinear effects important for $P_L > 2TW$
Relativistic self-focussing: Geometric optics

Consider laser beam with a radial profile

\[ a(r) = a_0 \exp(-r^2/2\sigma_L^2), \]

spot size \( \sigma_0 \) just inside a region of uniform, underdense plasma, see Fig. 7.

Figure 7: a) diffraction, b) self-focusing
Beam spreading due to diffraction will be cancelled by self-focusing effects if $\theta = \alpha$, (Exercise!),

$$a_0^2 \left( \frac{\omega_p \sigma_L}{c} \right)^2 \geq 8. \quad (27)$$

This represents a power threshold, since the laser power $P_L \propto a_0^2 \sigma_L^2$. In numbers:

$$P_L > 9 \left( \frac{\omega}{\omega_p} \right)^2 \text{ GW}$$

This is $\sim \times 2$ too low because we didn’t take beam profile into account.
Focussing threshold – practical units
Litvak, 1970; Max et al. 1974, Sprangle et al. 1988

Relation between laser power and critical power:

\[ P_L = \left( \frac{m \omega c}{e} \right)^2 \left( \frac{c}{\omega_p} \right)^2 c \varepsilon_0 \frac{1}{2} \int_0^\infty 2\pi r a^2(r) dr \]

\[ = \frac{1}{2} \left( \frac{m}{e} \right)^2 c^5 \varepsilon_0 \left( \frac{\omega}{\omega_p} \right)^2 \tilde{P} \]

\[ \simeq 0.35 \left( \frac{\omega}{\omega_p} \right)^2 \tilde{P} \text{ GW.} \]

The critical power \( \tilde{P}_c = 16\pi \) thus corresponds to:

Power threshold for relativistic self-focussing

\[ P_c \simeq 17.5 \left( \frac{\omega}{\omega_p} \right)^2 \text{ GW,} \quad (28) \]
Focussing threshold – example

Critical power

\[ P_c \approx 17.5 \left( \frac{\omega}{\omega_p} \right)^2 \text{GW}, \quad (29) \]

Example

\[ \lambda_L = 0.8 \mu m, \quad n_e = 10^{19} \text{ cm}^{-3} \]

\[ \Rightarrow \frac{n_e}{n_c} = \left( \frac{\omega_p}{\omega} \right)^2 = \frac{10^{19}}{1.6 \times 10^{21}} = 6 \times 10^{-3} \]

\[ \Rightarrow P_c = 2.6 \text{TW} \]
Transverse plasma response

Cigar-shaped pulse: take $\nabla = \nabla_\perp$, and apply Poisson’s equation ($\phi$ and $n$ normalized as before)

$$\nabla_\perp^2 \phi = k_p^2 (n - 1),$$

to obtain density perturbation:

$$n = 1 + k_p^{-2} \nabla_\perp^2 \gamma. \quad (30)$$
Cavitation condition

Consider Gaussian pulse profile \( a(r) = a_0 \exp(-r^2/2\sigma^2) \). After time-averaging over the laser period, the density depression term in cylindrical coordinates can be written:

\[
\nabla^2 \gamma = \frac{1}{4\gamma} \nabla^2 a^2 = \frac{1}{4\gamma} \frac{4a_0^2}{\sigma^2} \left( \frac{r^2}{\sigma^2} - 1 \right) \exp(-r^2/\sigma^2).
\]

The deepest depression is on the laser axis at \( r = 0 \), giving

\( I_{18} \lambda^2 \mu > 1/20 n_{18} \sigma^2 \mu \), (31) – quite easily fulfilled with TW lasers.
Cavitation condition

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The deepest depression is on the laser axis at $r = 0$, giving

Cavitation condition, $(n = 0)$:

$$I_{18\lambda^2} \chi^2_{\mu} > \frac{1}{20} n_{18} \sigma^2_{\mu}, \quad (31)$$

– quite easily fulfilled with TW lasers.
Relativistic beam propagation

Numerical solution of NLSE with an initial radial Gaussian beam profile with \( \sigma_0 = 7.5 \, \mu \text{m} \) and pump strengths \( a_0 \). Beam powers \( P/P_c \): 0.5, 1.0 and 3.0 respectively.
Plasma (Langmuir) wave propagation

Without the laser driving term \( A_y = 0 \), Eq.(23) describes linear plasma oscillations with solutions

\[
E_x = E_x^0 \sin(\omega t),
\]

\[
-\omega^2 + \omega_p^2 = 0 \tag{32}
\]

The linear eigenmode of a plasma has \( \omega = \omega_p \).

Including finite temperature \( T_e > 0 \) yields the Bohm-Gross relation:

\[
\omega^2 = \omega_p^2 + 3 v_t^2 k^2 \tag{33}
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\[ \omega^2 = \omega_p^2 + 3v_t^2k^2. \] \hspace{1cm} (33)
Numerical solutions – linear Langmuir wave

Numerical integration of the electrostatic wave equation on slide 1 for \( \nu_{\text{max}}/c = 0.2 \)

NB: electric field and density 90° out of phase
Numerical solutions – *nonlinear* Langmuir waves

Parameters here: a) phase velocity $v_p/c \simeq 1$ and $v_m/c = 0.9$; b) $v_p/c = 0.6$, $v_m/c = 0.55$

Typical features: i) sawtooth electric field; ii) spiked density; iii) *lengthening* of the oscillation period by factor $\gamma$. 

![Graphs showing typical features](image)
Maximum field amplitude - wave-breaking limit

For relativistic phase velocities, find

\[ E_{\text{max}} \sim m\omega_p c/e \]

Maximum field amplitude - wave-breaking limit

For relativistic phase velocities, find

\[ E_{\text{max}} \sim m\omega_p c/e \]


Example

\[ m_e = 9.1 \times 10^{-28} \text{g} \]
\[ c = 3 \times 10^{10} \text{cms}^{-1} \]
\[ \omega_p = 5.6 \times 10^{4}(n_e/cm^{-3})^{1/2} \]
\[ e = 4.8 \times 10^{-10} \text{statcoulomb} \]

\[ E_p \sim 4 \times 10^8 \left( \frac{n_e}{10^{18} \text{cm}^{-3}} \right)^{1/2} \text{V m}^{-1} \]
Wakefield excitation
Resonance condition

The amplitude of the longitudinal oscillation will be enhanced if the pulse length is roughly matched to the plasma period:

$$\tau_L \sim \omega_p^{-1}.$$
Resonance condition

The amplitude of the longitudinal oscillation will be enhanced if the pulse length is roughly matched to the plasma period:

\[ \tau_L \approx \omega_p^{-1}. \]

Example

What plasma density do we need to match a 100 fs pulse?

\[ \omega_p \approx 5 \times 10^4 n_e^{1/2} \text{ s}^{-1} \]
Resonance condition

The amplitude of the longitudinal oscillation will be enhanced if the pulse length is roughly matched to the plasma period:

\[ \tau_L \approx \omega_p^{-1}. \]

Example

What plasma density do we need to match a 100 fs pulse?

\[ \omega_p \approx 5 \times 10^4 n_e^{1/2} \text{ s}^{-1} \]

Matching condition:

\[ n_e \approx 4 \times 10^{14} \tau_{\text{ps}}^{-2} \text{ cm}^{-3} \]

For 100 fs, need \( n_e = 4 \times 10^{16} \text{ cm}^{-3}. \)
Numerical solution: small laser amplitude
Numerical solution: resonance condition (small amplitudes)
Numerical solution: large laser amplitude
Wake amplitude scaling in nonlinear regime
Murusidze & Berzhiani, 1990

Analytical solution possible for a square pump in the limit $\beta_g \to 1$

⇒ Scaling of the wake-variable maxima:

$$\phi_{\text{max}} \sim \gamma_\perp^2 - 1$$

$$E_{\text{max}} \sim \frac{\gamma_\perp^2 - 1}{\gamma_\perp}$$

$$p_{\text{max}} \sim (\gamma u)_{\text{max}} = \frac{\gamma_\perp^4 - 1}{2\gamma_\perp^2}$$

(34)

where $\gamma_\perp = (1 + a^2)^{1/2}$ as on p. ??.
2D wakefield excitation
Electron acceleration by wakefields

- Conventional synchrotrons and LINACS operate with field gradients limited to around 100 MVm\(^{-1}\).
- Plasma is already ionized; can theoretically sustain a field 10\(^4\) times larger, given by:

\[
E_p = \frac{m_e c \omega_p}{e} \varepsilon \\
\simeq n_{18}^{1/2} \varepsilon \text{ GV cm}^{-1},
\]

(35)

where \(n_{18}\) is the electron density in units of 10\(^{18}\) cm\(^{-3}\).
Laser-driven wakefields must propagate with velocities approaching the speed of light ($v_p = v_g < c$). Plasma wave has a phase velocity:

$$v_p = c \left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^{\frac{1}{2}} \simeq c \left(1 - \frac{1}{2\gamma_p^2}\right), \quad (36)$$

where $\gamma_p = \omega_o^2/\omega_p^2$. 

Tajima & Dawson, 1979
Acceleration length

A relativistic electron \((v \approx c)\) trapped in such a wave will be accelerated over at most half a wavelength in the wave-frame, after which it starts to be decelerated.

Effective acceleration length:

\[
L_a = \frac{\lambda_p c}{2(c - v_p)} \approx \lambda_p \gamma_p^2
\]

\[
= \frac{\omega^2}{\omega_p^2} \lambda_p
\]

\[
\approx 3.2 n_{18}^{-3/2} \lambda_{\mu m}^{-2} \text{ cm.} \quad (37)
\]
Combine Eq. (35) and Eq. (37) to obtain the maximum energy gain:

\[ \Delta U = eE_p L_a \]

\[ = e \left( \frac{m \omega_p c}{e} \right) \varepsilon \frac{\omega^2}{\omega_p^2} \frac{2 \pi c}{\omega_p} \]

\[ = 2 \pi \left( \frac{\omega}{\omega_p} \right)^2 \varepsilon mc^2 \]

\[ \approx 3.2 n_{18}^{-1} \lambda_{\mu m}^{-2} \text{GeV.} \quad (38) \]
Limiting factors

In principle, TW laser is capable of accelerating an electron to 5 GeV in a distance of 5 cm through a plasma with density $10^{18}$ cm$^{-3}$.

Spoiling factors:

- Diffraction: typically have $Z_R \ll L_a$, so some means of **guiding** the laser beam over the dephasing length is essential
- Propagation instabilities – beam break-up: modulation; hosing; Raman
Bench-Top Particle Accelerators

Standard electron acceleration in fast plasma wave (recap):

- Acceleration length

\[ L_a \sim 3.2 \, n_{18}^{-3/2} \lambda_{\mu m}^{-2} \text{ cm} \]

- Energy gain

\[ \Delta U \sim 3.2 \, n_{18}^{-1} \lambda_{\mu m}^{-2} \text{ GeV} \]
Acceleration mechanisms

Large variety of attributed acceleration mechanisms in experiments (long and short pulse):

- self-modulated
- forced-wave
- wave-breaking
- guided
- bubble-regime

GeV milestone reached September 2006 (LBL). More to come on Wed–Fri ....
What mechanisms are at work here?
Livingstone chart for laser-plasma electron accelerators

Figure 8: Open circles represent early beat-wave experiments; filled circles single pulse wakefield experiments; triangles quasi-*monoenergetic* electron beams
Summary of Lecture 2

Plasma response

Cold fluid equations

EM wave propagation

Plasma wave propagation

Wakefield excitation
Physics of High Intensity Laser Plasma Interactions
Part III: Interaction with Solids

20–25 June 2011  |  Paul Gibbon
Lecture 3: Interaction with Solids

Short pulse interaction scenarios

Collisional Absorption
  Normal skin effect

Collisionless Absorption
  Resonance absorption
  Brunel model

Hot Electron Generation
  Scaling
  Measurements of hot electron temperature

Ion acceleration
  Mechanisms
  Sheath model
  Hole boring
  Light sail
Short pulse vs. long pulse interactions

Long-pulse interaction physics (ICF – ns lasers):

- Collisional heating and creation of long scale-length plasmas
- Laser reflected at critical density surface
- Fast (keV) particles produced at ‘high’ intensities ($10^{16}$ Wcm$^{-2}$)
Short pulse vs. long pulse interactions

Long-pulse interaction physics (ICF – ns lasers):
- Collisional heating and creation of long scale-length plasmas
- Laser reflected at critical density surface
- Fast (keV) particles produced at 'high' intensities \((10^{16} \text{ Wcm}^{-2})\)

Femtosecond pulses
- Pulse length typically \(<\) ion motion timescale (hydrodynamics)
- Huge intensity range \(10^7\)
- No single interaction model possible
Typical interaction scenario: I
Creation of critical surface

Comination of field and collisional ionization over the first few laser cycles rapidly creates a surface plasma layer with a density many times the critical density $n_c$.

$$\omega^2 = \frac{4\pi e^2 n_c}{m_e}, \quad (39)$$

where $e$ and $m_e$ are the electron charge and mass respectively.
**Typical interaction scenario: I**

**Creation of critical surface**

Comination of field and collisional ionization over the first few laser cycles rapidly creates a surface plasma layer with a density many times the critical density \( n_c \).

\[ \omega^2 = \frac{4\pi e^2 n_c}{m_e}, \]  

where \( e \) and \( m_e \) are the electron charge and mass respectively.

In practical units (see Eq. 15):

\[ n_c \approx 1.1 \times 10^{21} \left( \frac{\lambda}{\mu\text{m}} \right)^{-2} \text{ cm}^{-3}. \]  

(40)
Interaction scenario: II

Ionization degree

Example

Al has 3 valence electrons; 6 more can be released for a few hundred eV. The electron density is given by:

\[ n_e = Z^* n_i = \frac{Z^* N_A \rho}{A}. \]  

(41)

effective ion charge: \( Z^* = 9 \)

atomic number: \( A = 26 \)

Avogadro number: \( N_A = 6.02 \times 10^{23} \)

mass density: \( \rho = \rho_{\text{solid}} = 1.9 \text{ g cm}^{-3} \)

electron density: \( n_e = 4 \times 10^{23} \text{ cm}^{-3} \)

density contrast (1 \( \mu \text{m} \)): \( n_e/n_c \simeq 400 \)
### Interaction scenario: II

#### Ionization degree

**Example**

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\[
 n_e = Z^* n_i = \frac{Z^* N_A \rho}{A}. 
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(41)

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
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<tbody>
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<tr>
<td>Density contrast (1 ( \mu \text{m} ))</td>
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</tr>
</tbody>
</table>
Target is heated via electron-ion collisions to 10s or 100s of eV depending on the laser intensity. The plasma pressure created during heating causes ion blow-off (ablation) at the sound speed:

\[
c_s = \left( \frac{Z^* k_B T_e}{m_i} \right)^{1/2}
\]

\[
\simeq 3.1 \times 10^7 \left( \frac{T_e}{\text{keV}} \right)^{1/2} \left( \frac{Z^*}{A} \right)^{1/2} \text{ cm s}^{-1}, \quad (42)
\]

where \( k_B \) is the Boltzmann constant, \( T_e \) the electron temperature and \( m_i \) the ion mass.
Interaction scenario: IV

Expansion

Because of ion ablation, density profile formed is *exponential* with scale-length:

\[ L = c_s \tau L \]

\[ \approx 3 \left( \frac{T_e}{\text{keV}} \right)^{1/2} \left( \frac{Z^*}{A} \right)^{1/2} \tau_{fs} \text{ Å}. \]  

(43)
Because of ion ablation, density profile formed is exponential with scale-length:

\[
L = c_s \tau_L
\]

\[
\simeq 3 \left( \frac{T_e}{\text{keV}} \right)^{1/2} \left( \frac{Z^*}{A} \right)^{1/2} \tau_{\text{fs}} \text{Å}. \tag{43}
\]

**Example**

100 fs Ti:sapphire pulse on Al foil heats the target to a few hundred eV → plasma with scale-length \( L/\lambda = 0.01–0.1 \). (cf: 100-1000 for ICF plasmas).
Collisional absorption

Modelled using **Helmholtz wave equations**: standard method for electromagnetic wave propagation in an inhomogeneous plasma – see books by Ginzburg, Kruer.

Start from Maxwell’s equations with small field amplitudes and a non-relativistic fluid response including collisional damping:

\[
\frac{m}{\partial t} \frac{\partial \mathbf{v}}{\partial t} = -e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - m\nu_{ei} \mathbf{v},
\]

where \( \nu_{ei} \) is the electron-ion collision frequency.
Collisional absorption


Start from Maxwell’s equations with small field amplitudes and a non-relativistic fluid response including collisional damping:

\[ m \frac{\partial \mathbf{v}}{\partial t} = -e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - m\nu_{ei} \mathbf{v}, \]

where \( \nu_{ei} \) is the electron-ion collision frequency.

Physically arises from binary collisions, resulting in a frictional drag on the electron motion.
Electron-ion collisional frequency
Spitzer-Härm

Collision rate:

\[ \nu_{ei} = \frac{4(2\pi)^{1/2}}{3} \frac{n_e Z e^4}{m^2 v_{te}^3} \ln \Lambda \]

\[ \simeq 2.91 \times 10^{-6} Z n_e T_e^{-3/2} \ln \Lambda \text{ s}^{-1}. \] (45)

\( Z = \) number of free electrons per atom
\( n_e = \) electron density in cm\(^{-3}\)
\( T_e = \) temperature in eV

\( \ln \Lambda \) is the Coulomb logarithm, with usual limits, \( b_{\text{min}} \) and \( b_{\text{max}} \), of the electron-ion scattering cross-section.
Coulomb logarithm

Limits are determined by the classical distance of closest approach and the Debye length respectively, so that:

\[ \Lambda = \frac{b_{\text{max}}}{b_{\text{min}}} = \lambda_D \cdot \frac{k_B T_e}{Z e^2} = \frac{9 N_D}{Z}, \quad (46) \]

where

\[ \lambda_D = \left( \frac{k_B T_e}{4\pi n_e e^2} \right)^{1/2} = \frac{v_{\text{te}}}{\omega_p}, \quad (47) \]

and

\[ N_D = \frac{4\pi}{3} \lambda_D^3 n_e \]

is the number of particles in a Debye sphere.
Absorption in steep density profiles: skin effect

Density profile can be approximated by a Heaviside step function:

\[ n_0(x) = n_0 \Theta(x), \]

\(\Theta(x)\) is the Heaviside step function.
Absorption in steep density profiles: skin effect

Density profile can be approximated by a Heaviside step function:

$$n_0(x) = n_0 \Theta(x),$$

giving a dielectric constant (cf: dispersion relation Eq. 24):

$$\varepsilon(x) \equiv \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2(1 + i\nu_{ei}/\omega)} \Theta(x). \quad (48)$$
Solution for laser field I

For normally incident light, the transverse electric field has the solution

\[ E_z = \begin{cases} 
2E_0 \sin(kx \cos \theta + \phi), & x < 0 \\
E(0) \exp(-x/l_s), & x \geq 0 
\end{cases} \quad (49) \]

where \( l_s \simeq c/\omega_p \) is the collisionless skin-depth, \( k = \omega/c \), \( E_0 \) is the amplitude of the laser field and \( \phi \) a phase factor.
Solution for laser field I

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\end{cases} \quad (49) \]

where \( l_s \simeq c/\omega_p \) is the collisionless skin-depth, \( k = \omega/c \), \( E_0 \) is the amplitude of the laser field and \( \phi \) a phase factor.

Matching vacuum and solid solutions at the boundary \( x = 0 \) gives:

\[ E(0) = 2E_0 \frac{\omega}{\omega_p} \cos \theta \]

\[ \tan \phi = -l_s \frac{\omega}{c} \cos \theta. \]
Solution for laser field I

\[ E_z^2(0) = 2E_0^2 \frac{n_c}{n_0} \]

Field intensity vs. \( kx \)

The figure shows the field intensity as a function of \( kx \), with two curves representing different components of the field. The equation \( E_z^2(0) = 2E_0^2 \frac{n_c}{n_0} \) is indicated on the graph, illustrating the relationship between the field intensity at the origin and the material properties.
Reflectivity: Drude model

Example

\[ \text{Al: } Z^* = 3, \quad n_e \simeq 2 \times 10^{23} \text{ cm}^{-3}, \quad \lambda_L = 0.8 \mu\text{m}, \quad n_e/n_c \simeq 100. \]

Figure 9: Angular absorption for a step-profile with \( n_e/n_c = 100 \) and \( \nu/\omega = 5 \) calculated analytically from the Fresnel equations (solid) and numerically from the Helmholtz wave equations (dotted).
Collisional frequency turn-off
quiver velocity correction

Effective collision frequency reduced by quiver motion in laser field

\[ \nu_{\text{eff}} \simeq \nu_{ei} \frac{v_{te}^3}{(v_{os}^2 + v_{te}^2)^{3/2}}. \quad (50) \]
Effective collision frequency reduced by quiver motion in laser field

\[ \nu_{\text{eff}} \approx \nu_{ei} \frac{v_{te}^3}{(v_{\text{os}}^2 + v_{te}^2)^{3/2}}. \quad (50) \]

A temperature of 1 keV corresponds to a thermal velocity \( v_{te} \approx 0.05 \), so collisional absorption starts to turn off for irradiances \( I\lambda^2 \geq 10^{15} \text{ Wcm}^{-2} \mu\text{m}^2 \).
Collisionless absorption mechanisms

What other absorption mechanisms can couple laser energy to a hot, solid-density target?
Collisionless absorption mechanisms

What other absorption mechanisms can couple laser energy to a hot, solid-density target?

1. Resonance absorption \( \left( \frac{L}{\lambda_L} \gg \frac{v_{os}}{\omega} \right) \) – Denisov (1957)
2. Anomalous (collisionless) skin effect – Weibel (1967)
Collisionless absorption mechanisms

What other absorption mechanisms can couple laser energy to a hot, solid-density target?

1. Resonance absorption \( \left( \frac{L}{\lambda_L} \gg \frac{v_{os}}{\omega} \right) \) – Denisov (1957)
2. Anomalous (collisionless) skin effect – Weibel (1967)
3. 'Vacuum heating’ – Brunel (1987), Gibbon (1992)

All of these mechanisms will generate fast electrons with energies \( T_h \sim \text{keV–MeV} \).
Collisionless resonance absorption

Figure 10: Standard picture of resonance absorption: a $p$-polarized light wave tunnels through to the critical surface ($n_e = n_c$) and drives up a plasma wave. This is damped by particle trapping and wave breaking at high intensities.
Resonance absorption: Denisov function

Self-similar dependence on the parameter \( \xi = (kL)^{1/3} \sin \theta \), \( kL \gg 1 \).
Absorption rate for long scalelengths

To a good approximation,

$$\phi(\xi) \simeq 2.3\xi \exp(-2\xi^3/3),$$  \hspace{1cm} (51)

and the fractional absorption is given by:

$$\eta_{ra} = \frac{1}{2} \Phi^2(\xi).$$

Behavior nearly independent of the damping mechanism.
Kinetic simulation of resonance absorption
Particle-in-Cell

Figure 11: PIC simulation of resonance absorption for parameters \( \theta = 9^0, \nu_{os}/c = 0.07, L/\lambda = 5 (kL = 10\pi) \), and \( n^\text{max}_e/n_c = 1.5 \): a) Density profile and electric field normal to the target, b) electric field parallel to the target, c) particle momenta, d) laser magnetic field.
Vacuum heating: Brunel model

Resonance absorption not possible if oscillation amplitude exceeds the density scale length $L$, i.e. if $v_{os}/\omega > L$. 

Figure 12: Capacitor model of the Brunel heating mechanism.
Vacuum heating: Brunel model

Resonance absorption not possible if oscillation amplitude exceeds the density scale length $L$, i.e. if $v_{os}/\omega > L$.

Capacitor approximation: magnetic field of the wave is ignored; laser electric field $E_L$ has some component $E_d$ normal to the target surface.

Figure 12: Capacitor model of the Brunel heating mechanism.
Driving electric field

\[ E_d = 2E_L \sin \theta. \]  \hspace{1cm} (52)

Field pulls a sheet of electrons out to a distance \( \Delta x \) from its initial position. Surface number density of this sheet is \( \Sigma = n_e \Delta x \), so the electric field created between \( x = -\Delta x \) and \( x = 0 \) is

\[ \Delta E = 4\pi e \Sigma. \]

Equating this to the driving field and solve for \( \Sigma \):

\[ \Sigma = \frac{2E_L \sin \theta}{4\pi e}. \]  \hspace{1cm} (53)
When the charge sheet returns to its original position, it acquires velocity $v_d \simeq 2v_{os} \sin \theta$, where $v_{os}$ is the usual electron quiver velocity.

Assuming these electrons are all ‘lost’ to the solid, then the average energy density absorbed per laser cycle is given by:

$$
P_a = \sum_{\tau} \frac{mv_d^2}{2} \simeq \frac{1}{16\pi^2} \frac{e}{m\omega} E_d^3.
$$
Brunel model IV

Compare to the incoming laser power:

\[ P_L = cE_L^2 \cos \theta / 8\pi. \]

Substituting Eq. (52), we obtain the fractional absorption rate for the Brunel mechanism:

\[ \eta_a \equiv \frac{P_a}{P_L} = \frac{4}{\pi} a_0 \frac{\sin^3 \theta}{\cos \theta}, \]

(54)

where \( a_0 = v_{os}/c \).

Expect more absorption at large angles of incidence and higher laser irradiance, \( I\lambda^2 \propto a_0^2 \).
Two corrections to improve the model:

1. Account for the reduced driver field amplitude due to absorption, replacing Eq. (52) by

\[ E_d = \left[ 1 + (1 - \eta_a)^{1/2} \right] E_L \sin \theta. \]  

(55)

2. Relativistic return velocities: replace kinetic energy with

\[ U_k = (\gamma - 1)mc^2 \]
Vacuum heating
Fully relativistic model
Gibbon, Andreev & Platanov (2011)

Figure 13: Absorption fraction vs. angle and intensity
Typical signature of collisionless laser heating – bi-Maxwellian electron spectrum with characteristic temperatures $T_c$ and $T_h$. 

- $T_e \sim 5$ keV
- $T_h \sim 75$ keV
Short pulse $T_h$ scaling

- Simple Brunel model:
  \[ T_h^B = \frac{mv^2}{2} \simeq 3.7 I_{16} \lambda^2_\mu \]  
  \[ (56) \]

- EM PIC simulations:
  \[ T_h^{GB} \simeq 7 (I_{16} \lambda^2)^{1/3} \]  
  \[ (57) \]

- Relativistic $j \times B$ model (ponderomotive scaling):
  \[ T_h^W \simeq mc^2 (\gamma - 1) \]
  \[ \simeq 511 \left[ (1 + 0.73 I_{18} \lambda^2_\mu)^{1/2} - 1 \right] \text{keV}. \]  
  \[ (58) \]
Hot electron temperature: short pulses

Figure 14: Hot electron temperature measurements in femtosecond laser-solid experiments (squares) compared with various models: long pulse — Eq. (??); Brunel — Eq. (56); Gibbon & Bell — Eq. (57) and Wilks — Eq. (58).
Ion acceleration

Direct acceleration in laser field inefficient, since

\[
\frac{v_i}{c} \approx \frac{eE_L}{m_i \omega c} = \frac{m_i}{m_e} a_0 \leq \frac{a_0}{1836}
\]
Ion acceleration

Direct acceleration in laser field inefficient, since

\[
\frac{v_i}{c} \approx \frac{eE_L}{m_i\omega c} = \frac{m_i}{m_e} a_0 \leq \frac{a_0}{1836}
\]

Get relativistic ion energies for

\[a_0 \sim 2000\]

or \[I\lambda_L^2 \geq 5 \times 10^{24} \text{ Wcm}^{-2}\mu\text{m}^2\]
Ion acceleration

Direct acceleration in laser field inefficient, since

\[
\frac{v_i}{c} \approx \frac{eE_L}{m_i\omega c} = \frac{m_i}{m_e} a_0 \leq \frac{a_0}{1836}
\]

Get relativistic ion energies for

\[
a_0 \sim 2000
\]

or

\[
L^2 \lambda_L \geq 5 \times 10^{24} \text{ Wcm}^{-2} \mu\text{m}^2
\]

Therefore need means of transmitting laser energy to ions over many cycles \(\rightarrow\) exploit electrostatic field in plasma.
Mechanisms

1. Coulomb explosion: clusters; ponderomotive channelling in gas jets
Mechanisms

1. Coulomb explosion: clusters; ponderomotive channelling in gas jets

2. Electrostatic sheath formed by hot electron cloud (TNSA)
Mechanisms

1. Coulomb explosion: clusters; ponderomotive channelling in gas jets

2. Electrostatic sheath formed by hot electron cloud (TNSA)

3. Collisionless shock formation: hole boring
Mechanisms

1. Coulomb explosion: clusters; ponderomotive channelling in gas jets

2. Electrostatic sheath formed by hot electron cloud (TNSA)

3. Collisionless shock formation: hole boring

4. Light sail: radiation pressure on mass-limited target
Sheath model

Electrostatic plasma expansion into vacuum: ions initially at rest \( (n_i = n_{i0}) \), hot electrons described by Boltzmann distribution:

\[
n_e = n_{e0} \exp(e\phi / T_h)
\]

where \( n_{e0} = Zn_{i0} \), and \( \phi \) satisfies Poisson’s equation:

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} (n_e - Zn_i)
\]
Sheath model

Electrostatic plasma expansion into vacuum: ions initially at rest \( n_i = n_{i0} \), hot electrons described by Boltzmann distribution:

\[
    n_e = n_{e0} \exp\left(\frac{e\phi}{T_h}\right)
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where \( n_{e0} = Zn_{i0} \), and \( \phi \) satisfies Poisson’s equation:

\[
    \frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} (n_e - Zn_i)
\]

Ion expansion is described by continuity and momentum equations:

\[
\begin{align*}
    \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x} \right) n_i &= -n_i \frac{\partial v_i}{\partial x} \\
    m_i \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x} \right) v_i &= -Ze \frac{\partial \phi}{\partial x}
\end{align*}
\]

(59)
Self-similar solution

If the plasma stays quasineutral everywhere \((n_e \sim Zn_i)\), then Eqs. (59) have a self-similar solution in \(x/t\):

\[
Zn_i = n_{e0} \exp(-x/c_s t - 1)
\]

\[
v_i = c_s + x/t
\]

\[
e\phi = -T_e\left(\frac{x}{c_s t} + 1\right)
\]

where \(c_s = \sqrt{ZT_e/m_i}\) is the ion sound speed.
Self-similar solution

If the plasma stays quasineutral everywhere \((n_e \simeq Zn_i)\), then Eqs. (59) have a self-similar solution in \(x/t\):

\[
\begin{align*}
Zn_i &= n_{e0} \exp(-x/c_s t - 1) \\
v_i &= c_s + x/t \\
e\phi &= -Te\left(\frac{x}{c_s t} + 1\right)
\end{align*}
\]  
(60)

where \(c_s = \sqrt{ZT_e/m_i}\) is the ion sound speed. Max ion velocity is

\[
v_f = 2c_s \log(\tau + \sqrt{\tau^2 + 1})
\]  
(61)

where

\[
\tau = \frac{\omega_{pi} t}{\sqrt{2e}}; \quad \omega_{pi} = \left(\frac{Z^2 e^2 n_{i0}}{\varepsilon_0 m_i}\right)^{1/2}
\]
Energy spectrum

Ion energy spectrum is given by:

\[
\frac{dN}{dU} = \frac{n_{i0}c_s t}{(2UU_0)^{1/2}} e^{-(2UU_0)^{1/2}}
\]  \hspace{1cm} (62)

where \( U_0 = Zk_B T_h \).

Figure 15: Ion energy spectrum from Mora expansion model
Hole boring

On the front side of an overdense plasma, the *ponderomotive force* will displace and compress electrons into the target, creating an electrostatic field acting on the ions.

\[
u_i = 2 \frac{u_s}{2} \left( \frac{I_{lm}}{m_i c_n e} \right)^{1/2} = 2 \sqrt{Z A m_e m_p n_c} (63)
\]

This leads to a quasi-monoenergetic component in the ion spectrum. See: Denavit, PRL (1992); Wilks, PRL (1992); Macchi, PRL (2005).

Hole boring

On the front side of an overdense plasma, the *ponderomotive force* will displace and compress electrons into the target, creating an electrostatic field acting on the ions.

Momentum balance across the collisionless, electrostatic shock thus formed implies:

\[
    u_i = 2u_s = 2 \left( \frac{I_L}{m_i n_i c} \right)^{1/2} = 2 \sqrt{\frac{Z m_e n_c}{A m_p n_e}} \tag{63}
\]

where \( u_s \) is the velocity of the shock front.
Hole boring

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\[ u_i = 2u_s = 2 \left( \frac{I_L}{m_i n_i c} \right)^{1/2} = 2 \sqrt{\frac{Z}{A} \frac{m_e n_c}{m_p n_e}} \]  \hspace{1cm} (63)

where \( u_s \) is the velocity of the shock front. This leads to a quasi-monoenergetic component in the ion spectrum. See: Denavit, PRL (1992); Wilks, PRL (1992); Macchi, PRL (2005) Relativistic HB formula: Robinson, PPCF (2009).
Light sail acceleration

A mass-limited target, such as a nm-thick foil, allows nearly complete displacement of electrons, thereby maximizing the ES field. A simple capacitor model suffices to determine the threshold intensity for this scenario.

\[ \Delta E = \varepsilon_0 n e d \]

This is balanced by the net laser field at the surface, \( 2E_L = 2m_0e\omega_{ca}/e \), leading to the matching condition:

\[ a_0 \approx \pi n e n_c d \lambda_L \]

Under these conditions, find max ion energy \( U_i \sim t_1/3 \) – see Esirkepov, PRL (2004).
Light sail acceleration

A mass-limited target, such as a nm-thick foil, allows nearly complete displacement of electrons, thereby maximizing the ES field. A simple capacitor model suffices to determine the threshold intensity for this scenario.

The charge separation field of a foil with thickness $d$ is:

$$\Delta E = \frac{e}{\varepsilon_0} n_e d$$

This is balanced by the net laser field at the surface, $2E_L = 2m_e \omega c a_0 / e$, leading to the matching condition:

$$a_0 \simeq \pi \frac{n_e d}{n_c \lambda_L}$$

(64)
Light sail acceleration

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(64)

Under these conditions, find max ion energy $U_i \sim t^{1/3} \quad \text{– see Esirkepov, PRL (2004)}. $
Summary of Lecture 3

- Short pulse interaction scenarios
- Collisional Absorption
- Collisionless Absorption
- Hot Electron Generation
- Ion acceleration
Physics of High Intensity Laser Plasma Interactions
Part IV: Numerical Simulation of Laser-Plasma Interactions

20–25 June 2011 | Paul Gibbon
Lecture 4: Numerical Simulation of Laser-Plasma Interactions

Plasma models
   Classification
   Hierarchy

Hydrodynamics
   Thermal transport
   Single fluid model
   Numerical scheme

Particle-in-Cell Codes
   Vlasov equation
   Particle pusher
   Field solver
   Algorithm
   3D codes
Plasma classification

Figure 16: Plasma classification in the density-temperature plane.
Plasma model hierarchy

Figure 17: Physical basis of common plasma models and corresponding numerical approaches.
Numerical modelling of laser-plasma interactions

Two types of modelling dominate LPI:

1. Hydrodynamic modelling to follow the macroscopic, dynamic behavior of the plasma, including external electric and magnetic fields, and heating by laser or particle beams: ps-ns timescale. Good for prepulse modelling.

2. Kinetic modelling for situations in which 'non-Maxwellian' particle distributions $f(\alpha)$ have to be determined self-consistently – this is the method of choice for laser-particle accelerator schemes!
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Figure 18: Building blocks of a laser-plasma hydro-code.
Heating rate of plasma slab

If penetration depth of the heat wave $l_h < l_s = c/\omega_p$ (skin depth), then the thermal transport can be neglected.
**Heating rate of plasma slab**

If penetration depth of the heat wave \( l_h < l_s = c/\omega_p \) (skin depth), then the thermal transport can be neglected.

Volume heated simultaneously: \( V \simeq l_s \pi \sigma_L^2 \).
Heating rate of plasma slab

If penetration depth of the heat wave \( l_h < l_s = c/\omega_p \) (skin depth), then the thermal transport can be neglected.

Volume heated simultaneously: \( V \simeq l_s \pi \sigma_L^2 \).

Setting \( \epsilon = \frac{3}{2} n_e k_B T_e \) and \( \nabla \cdot \Phi_a \sim \Phi_a / l_s \), have

\[
\frac{dT_e}{dt} \simeq \frac{\Phi_a}{n_e l_s}, \quad (65)
\]

heat rate:

\[
\frac{d}{dt} (k_B T_e) \simeq 4 \frac{\Phi_a}{W \text{cm}^{-2}} \left( \frac{n_e}{\text{cm}^{-3}} \right)^{-1} \left( \frac{l_s}{\text{cm}} \right)^{-1} \text{keV fs}^{-1}. \quad (66)
\]
Thermal transport

Energy transport equation for a collisional plasma:

\[
\frac{\partial \epsilon}{\partial t} + \nabla \cdot (q + \Phi_a) = 0,
\]  

(67)

where \( \epsilon \) is the energy density, \( q \) is the heat flow and \( \Phi_a = \eta_a \Phi_L \) is the absorbed laser flux.
Thermal transport

Energy transport equation for a collisional plasma:

\[
\frac{\partial \epsilon}{\partial t} + \nabla \cdot (q + \Phi_a) = 0, \tag{67}
\]

where \( \epsilon \) is the energy density, \( q \) is the heat flow and \( \Phi_a = \eta_a \Phi_L \) is the absorbed laser flux.

Huge temperature gradients are generated after a few fs: heat is carried away from the surface into the colder target material according to Eq. (67). For ideal plasmas, we write

\[
\epsilon = \frac{3}{2} n_e k_B T_e
\]

as before, and

\[
q(x) = -\kappa_e \frac{\partial T_e}{\partial x}, \tag{68}
\]

which is the usual Spitzer-Härm heat-flow.
Spitzer-Härm heat-flow

Substituting for $\epsilon$ and $q$ in Eq. (67) and restricting ourselves to 1D by letting $\nabla = (\partial/\partial x, 0, 0)$, gives a diffusion equation for $T_e$:

$$
\frac{3}{2} n_en_B \frac{\partial T_e}{\partial t} = \frac{\partial}{\partial x} \left( \kappa_e \frac{\partial T_e}{\partial x} \right) + \frac{\partial \Phi_L}{\partial x}.
$$

$\kappa_e$ is known as the Spitzer thermal conductivity and is given by:

$$
\kappa_e = 32 \left( \frac{2}{\pi} \right)^{1/2} \frac{n_e}{\nu_0 m^{5/2}} T_e^{5/2},
$$

where

$$
\nu_0 = \frac{2\pi n_e Z e^4 \log \Lambda}{m^2}.
$$
Figure 19: Nonlinear heat-wave advancing into a semi-infinite, solid-density plasma. The curves are obtained from the numerical solution of the Spitzer heat flow equation for constant laser absorption at the target surface (left boundary).
Single-fluid approximation

Fluid model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (71)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) + \nabla P - f_p = 0 \quad (72)$$

$$\frac{\partial \epsilon_e}{\partial t} + \nabla \cdot \left[ \mathbf{u}(\epsilon_e + P_e) - \kappa_e \nabla T_e - \frac{Q_{ei}}{\gamma_e - 1} - \Phi_a \right] = 0 \quad (73)$$

$$\frac{\partial \epsilon_i}{\partial t} + \nabla \cdot \left[ \mathbf{u}(\epsilon_i + P_i) - \kappa_i \nabla T_i + \frac{Q_{ei}}{\gamma_i - 1} \right] = 0, \quad (74)$$
Notes

- Average fluid density $\rho$ and velocity $u$ defined thus:

$$\rho \equiv n_i M + n_e m \simeq n_i M,$$

$$u \equiv \frac{1}{\rho} \left( n_i Mu_i + n_e Mu_e \right) \simeq u_i.$$

- The energy density $\epsilon_{\alpha}$ is the sum of internal and kinetic fluid energies:

$$\epsilon_{\alpha} = \frac{P_{\alpha}}{\gamma_{\alpha} - 1} + \frac{1}{2} \rho u^2,$$

with $\gamma_{\alpha}$ defined as the number of degrees of freedom.

- $Q_{ei}$ is the electron-ion equipartition rate

$$Q_{ei} = \frac{2m n_e k_B (T_e - T_i)}{M} \frac{1}{\tau_{ei}}$$

(75)

- $\tau_{ei} = \nu_{ei}^{-1}$ is the electron-ion collision time.

- Coupling to the laser $\Phi_a$ and $f_p$. 

Numerical Simulation of Laser-Plasma Interactions
Hydrodynamics
Single fluid model
Example: laser prepulse heating solid Al target

Figure 20: Density (solid line) and temperature (dashed line) profiles from two interactions with pulse durations of a) 100 fs and b) 100ps. The initially cold Al target extends to $x = 1 \, \mu m$. a) Snapshots 0 fs, 300 fs and 5 ps after the peak of the pulse; in b) snapshots are at -150 ps, -100 ps and 0 ps.
Kinetic plasma simulation

Simplest kinetic description of a plasma uses single-particle velocity distribution function $f(r, v)$, evolving according to the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + q\left(E + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial p} = 0. \tag{76}$$

- Distribution function $f(r, v)$ is 6-dimensional – direct solution of Eq. (76) generally impractical.
- Even 1D spatial geometry still needs 2 or 3 velocity components to couple to Maxwell’s equations, ie: 3- or 4-dimensional code.
Particle-in-Cell simulation

Important method developed in the 1960s is the so-called Particle-in-Cell (PIC) technique. Distribution function is represented instead by a large number of discrete *macro-particles*, each carrying a fixed charge $q_i$ and mass $m_i$.

Figure 21: Relationship between a) Vlasov and b) PIC approaches.
Particle pusher

Geometry for simplified Lorentz equation (see Eq. 16):
\[ \mathbf{E} = (0, E_y, 0), \; \mathbf{B} = (0, 0, B_z): \]

1/2-acceleration:

\[ p_x^- = p_x^{n-\frac{1}{2}}; \; p_y^- = p_y^{n-\frac{1}{2}} + \frac{\Delta t}{2} E_y^n \]

rotation:

\[ \gamma^n = \left(1 + (p_x^-)^2 + (p_y^-)^2\right)^{1/2}; \quad t = \frac{\Delta t}{2} \frac{B_z^n}{\gamma^n}; \quad s = \frac{2t}{1 + t^2} \]

\[ p_x' = p_x^- + p_y^- t \]

\[ p_y^+ = p_y^- - p_x' s \]  \hspace{1cm} (77)

\[ p_x^+ = p_x' + p_x^+ t \]

1/2-acceleration:

\[ p_x^{n+\frac{1}{2}} = p_x^+; \; p_y^{n+\frac{1}{2}} = p_y^+ + \frac{\Delta t}{2} E_y^n. \]
Density & current gather

The density and current sources needed to integrate Maxwell’s equations are obtained by mapping the local particle positions and velocities onto a grid as follows:

\[
\rho(r) = \sum_j q_j S(r_j - r),
\]

\[
J(r) = \sum_j q_j v_j S(r_j - r), \quad j = 1 \ldots N_{\text{cell}} \tag{78}
\]

where \( S(r_j - r) \) is a function describing the effective shape of the particles. Usually it is sufficient to use a linear weighting for \( S \) — the ‘Cloud-in-Cell’ scheme — although other more accurate methods (eg: cubic spline) are also possible.
Once $\rho(r)$ and $\mathbf{J}(r)$ are defined at the grid points, we can proceed to solve Maxwell’s equations to obtain the new electric and magnetic fields. In 2D geometry, can define:

\[
F^+ = \frac{1}{2}(E_y + B_z)
\]
\[
F^- = \frac{1}{2}(E_y - B_z)
\]

Then Faraday and Ampere laws reduce to (normalised units):

\[
F^+_{i+1}(n+1) = F_i(n) - \frac{\Delta t}{2} J_{y,i+1/2}^{n+1/2} \quad \text{(forward)}
\]
\[
F^-_i(n+1) = F_{i+1}(n) - \frac{\Delta t}{2} J_{y,i+1}^{n+1/2} \quad \text{(backward)}
\]
The fields are then interpolated back to the particle positions so that we can go back to the particle push step Eq. (??) and complete the cycle, see Fig. 22.

Figure 22: Iteration steps of the particle-in-cell algorithm.
PIC codes for laser-plasma interaction studies

Because of its simplicity and ease of implementation, the PIC-scheme sketched above is currently one of the most important plasma simulation methods.

<table>
<thead>
<tr>
<th>Name</th>
<th>Authors</th>
<th>Group(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>OSIRIS</td>
<td>Mori, Silva</td>
<td>UCLA, IST Lisbon</td>
</tr>
<tr>
<td>VLPL</td>
<td>Pukhov</td>
<td>U. Düsseldorf</td>
</tr>
<tr>
<td>REMP</td>
<td>Esirkepov</td>
<td>Kyoto, Moscow</td>
</tr>
<tr>
<td>VPIC</td>
<td>Lin, Bowers</td>
<td>LANL</td>
</tr>
<tr>
<td>EPOCH</td>
<td>Arber, Bennett</td>
<td>U. Warwick (CCPP)</td>
</tr>
<tr>
<td>CALDER</td>
<td>Lefebvre</td>
<td>CEA</td>
</tr>
<tr>
<td>SPSC</td>
<td>Ruhl, Brömmel</td>
<td>LMU, FZ Jülich</td>
</tr>
</tbody>
</table>

Table 3: Parallel 3D, electromagnetic PIC codes
Summary of Lecture 4

Plasma models

Hydrodynamics

Particle-in-Cell Codes
Physics of High Intensity Laser Plasma Interactions
Part V: Tutorial on Particle-in-Cell Simulation

20–25 June 2011  |  Paul Gibbon
Lecture 5: Tutorial on Particle-in-Cell Simulation

The PIC code BOPS

Prerequisites

Installation

Running BOPS

Project I: Laser wakefield accelerator

Project II: Ion acceleration – TNSA vs. RPA
The PIC code BOPS

- BOPS is a one-and-three-halves (1 spatial, 3 velocity coordinates: 1D3V) particle-in-cell code originally created by Paul Gibbon and Tony Bell in the Plasma Physics Group of Imperial College, London.
- Based on standard algorithm for a 1D electromagnetic PIC code from Birdsall & Langdon.
- Optionally employs a Lorentz ‘boost’ along the target surface to mimic 2D, periodic-in-y geometry, with big savings in compute time.
- Applications: absorption, electron heating, high harmonic generation, ion acceleration
Prerequisites

- Current version: BOPS 3.4
- Download site: https://trac.version.fz-juelich.de/bops
- Compiler: sequential code written in Fortran 90 (Intel ifort or GNU’s gfortran)
- Run scripts provided are designed for generic Unix systems: Ubuntu, SuSE, RedHat and also MacOS.

**Linux and MAC users:**

Unpack the tar file (name may differ) with:

```bash
tar xvzf bops3.4.tar.gz
```

and `cd` to the installation directory `bops`. 
Windows users

1. First install CYGWIN (www.cygwin.com). This creates a fully-fledged Unix environment emulator under Windows. In addition to the default tools/packages offered during the installation you will also need:
   - Gnu compilers gcc, g77, g95 etc. (*devel* package)
   - make (*devel* package)
   - vi, emacs (*editors* package)
   - X11 libraries if you want to generate graphics directly under cygwin (eg using gnuplot)

If you forget anything first time, just click on the Cygwin installer icon to locate/update extra packages.

2. Download and unpack the bops3.4.tar.gz file with an archiving tool (eg PowerArchiver: www.powerarchiver.com). Put this in your ‘home’ directory under CYGWIN, e.g.:

   `C:\cygwin\home\<username>\`

3. Open a Cygwin terminal/shell and ‘cd’ to the bops directory.
Installation

The directory structure resulting from unpacking the tar files should look like this:

- src/ ... containing the fortran90 source code
- doc/ ... some documentation in html and ps
- tutorial/ ... scripts and files for current tutorial
- benchmarks/ ... additional sample scripts for running the code
- tools/ ... postprocessing tools

To compile:

Go to the source directory src, and edit the Makefile: adjust and tune the flags to match your machine type (FC=ifort or gfortran etc). Then do:

make
Running BOPS

Once the code has compiled, go back up to the base (or top) directory and enter the tutorial directory. Here you will find the run scripts (suffix .sh) which launch the example simulations. These can be edited directly or copied as needed.

To run from this directory, just type eg:

```
./wake.sh
```

Or if you prefer to stay in the top directory – adjust paths in wake.sh first:

```
tutorial/wake.sh
```

Further examples can be found in benchmarks
1. Edit and run the script: `./wake.sh`. Note that the input parameters are normalised to laser wavelength and wavenumber.

2. Look at the printed output to check the actual laser and plasma parameters (are they what you intended?)

### Plot files

These are in run directory (`foil_tnsa1`) in ASCII format and have a suffix `NN.xy`, where `NN` is the snapshot number. See `bops.oddata` for complete list.

- **ezsi**: EM field $E_z$ (S-polarized wave)
- **nenc**: Electron density
- **exsi**: Electrostatic field
- **pxxe**: Electron momenta ($x - p_x$ phase space)

3. Examine the field and particle phase space plots at successive time intervals (e.g.: $t=100, t=200$). The gnuplot script `wake.gp` should help you get started.

4. How can the plasma wave amplitude $|E_x|_{\text{max}}$ be optimised? Hint: try changing the electron density and/or pulse length.

5. Increase the laser amplitude to ($a_0 = 3$) and observe the difference in the electron phase space and plasma wave. How does the matching condition need to be altered in this regime? Why is the amplitude of the latter reduced towards the back of the wake?

6. (advanced) Set up a plasma equivalent to twice the dephasing length (Eq. 37) and determine the maximum energy of electrons trapped in the plasma wave. Can you beat the scaling predicted by Eq. 38?
Project II: Ion acceleration – TNSA vs. RPA

1 TNSA regime.
   a) The script `foil.sh` sets up a 2 μm ‘foil’ out of frozen hydrogen \((Z = A = 1)\), with density \(n_e/n_c = 36\). This is irradiated by a linearly polarized 50 fs pulse with intensity \(10^{19} \text{ Wcm}^{-2}\).

   Run the script and inspect the following plots at the snapshot times \((0, 50, 100, 150)\) fs:

<table>
<thead>
<tr>
<th>Plot files</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>nenc, ninc</code> Electron &amp; Ion densities</td>
</tr>
<tr>
<td><code>exdc</code> Cycle-averaged electrostatic field</td>
</tr>
<tr>
<td><code>pxxe, pxxi</code> Electron &amp; ion momenta</td>
</tr>
<tr>
<td><code>fuep, fuip</code> Electron, ion energy spectra</td>
</tr>
<tr>
<td><code>uinc, ubac</code> Incoming and outgoing wave energy</td>
</tr>
</tbody>
</table>

   b) Try varying, eg: pulse amplitude or duration and compare the scaling of the maximum ion energy against the theoretical prediction of Eq. (15).

2 HB regime.
   a) Copy the script to a new file (eg: `hb.sh`), switch the polarization from linear to circular (set `cpolzn='C'`), change the run directory – eg: `RUN=hb1` and rerun. (What phase space variables can you check the laser polarization with?) Compare the results to the TNSA case.

   b) To compare with HB theory, it is convient to specify a ‘square’ pulse profile. This can be done with, eg: `ilas=1, trise=5, tpulse=150`. Another important constraint in this case is to avoid numerical heating arising from an underresolved (cold) electron Debye length. Adjust the grid size/resolution to ensure that \(\Delta x < 0.5 \lambda_D\). Rerun and compare the ion shock velocity with Eq. (63).

3 Light sail regime.
   a) Use the matching condition Eq. (64) to determine the foil width (using the same laser and target material) for which the light sail regime is reached. Set `ilas=4` with `tpulse=50, tfall=5`, and ensure that the new foil is resolved by the grid. Create a new script with these parameters and rerun. NB: you may have to increase the parameter `uimax` for this case!

   b) Investigate the scaling behaviour of the monoenergetic ion feature with intensity, pulse length etc.

   c) Now repeat with a Gaussian or \(\sin^2\) pulse shape. Compare the hot electron spectrum and phase space with the flat-top case at early times: what causes this difference? What could you change to improve the far-field ion sail stability for realistic pulse forms?

   d) Now try the same thing with a carbon foil. Here, you will need higher (solid) number densities and multiply charged ions (see Eq. 41). What happens to the max. ion energy? How does the energy/nucleon compare with the hydrogen case?