Part V

Interaction with Underdense Plasmas - Nonlinear Wave Propagation and Wave-Breaking
5 Laser Propagation in Underdense Plasmas

Nonlinear Plane Waves in a Cold Plasma
Linear dispersion relations
Nonlinear plasma oscillations
Wavebreaking
Ionized gases: when is plasma response important?

- Simultaneous field ionization of many atoms produces a plasma with electron density $n_e$, temperature $T_e \sim 1 - 10$ eV.
- *Collective effects* important if
  \[ \omega_p \tau_L > 1 \]
- Example: $\tau_L = 100$ fs, $n_e = 10^{17}$ cm$^{-3}$ $\rightarrow \omega_p \tau_L = 1.8$
- Typical gas jets: $P \sim 1$ bar; $n_e = 10^{18} - 10^{19}$ cm$^{-3}$
- Underdense: $\omega^2 / \omega_p^2 = n_e / n_c \ll 1$; $n_c(1 \mu) = 10^{21}$ cm$^{-3}$
- Exploit plasma effects for: (short-wavelength) radiation; high electric/magnetic fields; nonlinear refractive properties
Lorentz-Maxwell equations

\[
\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla)p = -e(E + \frac{1}{c}\mathbf{v} \times \mathbf{B}), \quad (58)
\]

\[
\nabla \cdot \mathbf{E} = 4\pi e(n_0 - n_e), \quad (59)
\]

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (60)
\]

\[
\nabla \times \mathbf{B} = -\frac{4\pi}{c} en_e \mathbf{v} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad (61)
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (62)
\]

where \( p = \gamma m \mathbf{v} \) and \( \gamma = (1 + p^2/m^2c^2)^{1/2} \).
Nonlinear plane-wave solutions
Akhiezer & Polovin, 1956

Look for solutions of the form $f(\omega t - \mathbf{k} \cdot \mathbf{r})$, or $f(\tau)$, where $\tau = t - \mathbf{i} \cdot \mathbf{r} / v_p$, and $v_p = \omega / k$ is the phase velocity of the wave. Temporal and spatial differential operators become:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau},$$
$$\nabla \cdot = -\frac{i}{v_p} \frac{\partial}{\partial \tau},$$
$$\nabla \times = -\frac{i}{v_p} \frac{\partial}{\partial \tau} \times,$$

where $\mathbf{i} = \mathbf{k} / | \mathbf{k} |$ is a unit vector in the direction of wave propagation.
Simplified Maxwell-fluid equations
– but still nonlinear!

\[
\begin{align*}
\left( \frac{\mathbf{i} \cdot \mathbf{v}}{v_p} - 1 \right) \frac{dp}{dT} &= e\left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}\right), \\
-i \cdot \frac{d\mathbf{E}}{dT} &= 4\pi e v_p (n_0 - n_e), \\
\mathbf{B} &= \frac{c}{v_p} \mathbf{i} \times \mathbf{E} \\
-i \times \frac{d\mathbf{B}}{dT} &= -\frac{4\pi}{c} e v_p n_e \mathbf{v} + \frac{v_p}{c} \frac{d\mathbf{E}}{dT}, \\
i \cdot \frac{d\mathbf{B}}{dT} &= 0.
\end{align*}
\]

NB: \(i \cdot \mathbf{B} = \mathbf{E} \cdot \mathbf{B} = 0\) – B-field perpendicular to both the wave vector and E-field.
Now take the dot product of Ampere Eq. (66) with the direction vector $\mathbf{i}$:

$$\mathbf{i} \cdot \left(-\mathbf{i} \times \frac{d\mathbf{B}}{d\tau}\right) = \mathbf{i} \cdot \left(-\frac{4\pi}{c} e v_p n_e \mathbf{v} + \frac{v_p}{c} \frac{d\mathbf{E}}{d\tau}\right)$$

Eliminate $\mathbf{E}$ using Eq. (64) to give an equation for the density:

$$n_e = \frac{v_p n_0}{v_p - \mathbf{i} \cdot \mathbf{v}}.$$  \hspace{1cm} \text{(68)}$$
Similarly, the cross-product of the direction vector with Eq. (63) yields an explicit equation for $\mathbf{B}$, namely:

$$
\mathbf{B} = -\frac{c}{ev_p} \mathbf{i} \times \frac{d\mathbf{p}}{d\tau}.
$$

Likewise, taking $\mathbf{i} \times (66)$ and making use of Eq. (65), we obtain an equation for $d\mathbf{B}/d\tau$:

$$
\frac{d\mathbf{B}}{d\tau} = \frac{4\pi en_e}{\beta_p^2 - 1} \mathbf{i} \times \mathbf{v},
$$

where $\beta_p = v_p/c$. 
We can now eliminate $\mathbf{B}$ from the previous two equations by subtracting Eq. (70) from $d/d\tau(69)$, leaving a transverse wave equation

$$\mathbf{i} \times \frac{d^2 \mathbf{p}}{d\tau^2} + \frac{4\pi e^2 n_e \beta_p^2}{\beta_p^2 - 1} \mathbf{i} \times \mathbf{v} = 0.$$  (71)
Longitudinal wave equation

Longitudinal component of the fluid motion (eliminate $\mathbf{B}$ from $\mathbf{i}\cdot\text{Eq. (63)}$):

$$
\frac{d}{d\tau} \left[ \left( \frac{\mathbf{i} \cdot \mathbf{v}}{v_p} - 1 \right) \mathbf{i} \cdot \frac{d\mathbf{p}}{d\tau} \right] = \frac{4\pi e^2 v_p n_0 \mathbf{i} \cdot \mathbf{v}}{v_p - \mathbf{i} \cdot \mathbf{v}} \left( \frac{1}{v_p} \frac{d}{d\tau} \left[ \mathbf{v} \cdot \frac{d\mathbf{p}}{d\tau} - (\mathbf{i} \cdot \mathbf{v})(\mathbf{i} \cdot \frac{d\mathbf{p}}{d\tau}) \right] \right).
$$

(72)
To render equations (71) and (72) into something more tractable, we specify the wave vector $\mathbf{k}$ to be in the $x$-direction. Thus, we have

$$\mathbf{i} = \hat{x}, \quad \mathbf{i} \cdot \mathbf{p} = p_x \quad \text{and} \quad \mathbf{i} \times \mathbf{p} = (0, -p_z, p_y).$$

With these simplifications, and defining $\mathbf{u} = \mathbf{v}/c$, Eq. (68) becomes:

$$n_e = \frac{\beta_p n_0}{\beta_p - u_x}.$$  \hspace{1cm} (73)

where $\beta_p = v_p/c$, $u_x = v_x/c$.

Typical feature of nonlinear plasma waves: density becomes very large in regions where the fluid velocity approaches the phase velocity.
Simplified wave equations – transverse

Taking the $y$ and $z$ components of Eq. (71) and making use of Eq. (73) and the usual definition $\omega_p^2 = 4\pi e^2 n_0 / m$, gives us the coupled transverse wave equations:

\[
\frac{d^2 p_y}{d\tau^2} + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1} \frac{\beta_p u_y}{\beta_p - u_x} = 0, \tag{74}
\]

\[
\frac{d^2 p_z}{d\tau^2} + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1} \frac{\beta_p u_z}{\beta_p - u_x} = 0, \tag{75}
\]

where now $p_y$ and $p_z$ have been normalized to $mc$, so that $\mathbf{p} = \gamma \mathbf{u}$. 
Simplified wave equations – longitudinal

The longitudinal wave equation follows from Eq. (72), which on applying the same choice of wave vector, simplifies to:

$$\frac{d}{d\tau} \left[ (u_x - \beta_p) \frac{dp_x}{d\tau} + u_y \frac{dp_y}{d\tau} + u_z \frac{dp_z}{d\tau} \right] = \frac{\omega_p^2 \beta_p^2 u_x}{\beta_p - u_x}.$$  \hspace{1cm} (76)

Closed set of equations (74–76) for nonlinear plasma waves of arbitrary amplitude and fixed phase velocity $v_p$. 
Simplified wave equations – electric and magnetic fields

Once solved for \( p \), the corresponding electric and magnetic fields (normalized here to \( m\omega_p c/e \)) can be obtained from:

\[
E_x = -\frac{1}{\beta_p} \frac{d}{d(\omega_p \tau)} \left( \beta_p p_x - (1 + p^2)^{\frac{1}{2}} \right),
\]

\[
E_y = -\frac{dp_y}{d(\omega_p \tau)},
\]

\[
E_z = -\frac{dp_z}{d(\omega_p \tau)},
\]

\[
B_x = 0,
\]

\[
B_y = \frac{1}{\beta_p} \frac{dp_z}{d(\omega_p \tau)},
\]

\[
B_z = -\frac{1}{\beta_p} \frac{dp_y}{d(\omega_p \tau)}.
\]

Potential: \( \phi = \gamma - \beta_p p_x - 1. \)
Low amplitude limit – linear dispersion relation

Transverse equation:

\[
\frac{d^2 p_y}{d \tau^2} + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1} \frac{\beta_p u_y}{\beta_p - u_x} = 0
\]

Linearize equation:

- \( p_{x,y,z} \ll 1 \)
- \( u_x \ll \beta_p \)

\( \rightarrow \) neglect all terms \( O(u^2) \) and higher

Geometry: EM wave propagates along the \( x \)-axis:

\[ E_L = (0, E_y, 0), \quad B_L = (0, 0, B_z), \quad p_y = A_y. \]
Longitudinal and transverse wave components \textit{decouple}, so that we recover the linear wave equation

\[
\frac{d^2 u_y}{d\tau^2} + \frac{\omega_p^2 \beta_p^2 u_y}{\beta_p^2 - 1} = 0,
\]

(79)

Solution:

\[u_y = u_o e^{-i\omega \tau},\]

provided that

\[-\omega^2 + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1} = 0.\]

Or, since \(\beta_p = v_p/c = \omega/kc\), can rearrange to get:

\[\omega^2 = \omega_p^2 + c^2 k^2.\]

(80)
Dispersion relation - longitudinal (Langmuir) waves

Linearizing the longitudinal momentum equation (Eq. (76)) yields

$$-\beta_p \frac{d^2 p_x}{d\tau^2} - \omega_p^2 \beta_p u_x = 0,$$

or, since $\gamma \approx 1$,

$$\frac{d^2 u_x}{d\tau^2} + \omega_p^2 u_x = 0. \quad (81)$$

Solution: $u_x = u_{x0} e^{-i\omega \tau}$, with dispersion relation for longitudinal waves in the limit $T_e \to 0$.

$$\omega^2 = \omega_p^2.$$

Include finite temperature $T_e > 0$ – Bohm-Gross relation:

$$\omega^2 = \omega_p^2 + 3v_t^2 k^2. \quad (82)$$
Dispersion relations - graphical

\[ \omega^2 = \omega_p^2 + 3v_t^2 k^2 \]

\[ \omega^2 = \omega_p^2 + c^2 k^2 \]

\[ \omega = \omega_p \]
Nonlinear plasma oscillations
Noble, 1984

\[
\frac{d}{d\tau} \left[ (u_x - \beta_p) \frac{dp_x}{d\tau} + u_y \frac{dp_y}{d\tau} + u_z \frac{dp_z}{d\tau} \right] = \frac{\omega_p^2 \beta_p^2 u_x}{\beta_p - u_x}.
\]

Set \( p_y = p_z = 0; u_x = u \) in Eq. (76), which simplifies to

\[
\frac{d}{d\tau} \left[ (u - \beta_p) \frac{dp}{d\tau} \right] = \frac{\omega_p^2 \beta_p^2 u}{\beta_p - u}.
\]

Writing \( p = \gamma u = u/\sqrt{1 - u^2} \) and rearranging gives a 2nd order differential equation for the longitudinal velocity alone:

\[
\frac{d^2}{d\tau^2} \left[ \gamma(1 - \beta_p u) \right] = \frac{\omega_p^2 \beta_p^2 u}{\beta_p - u}.
\] (83)
Solution for longitudinal wave

This equation can be integrated once to give:

\[ \frac{1}{2} Y^2 = \beta_p^2 \omega_p^2 (\gamma_m - \gamma), \]

where

\[ Y = \frac{d}{d\tau} [\gamma(1 - \beta_p u)], \quad \gamma_m = (1 - u_m^2)^{-1/2}, \]

and \( u_m = (v/c)_{max} \) is the maximum oscillation velocity of the wave. The waveform can thus be determined from the solution of:

\[ \frac{d}{d\tau} [\gamma(1 - \beta_p u)] = \pm \sqrt{2} \omega_p \beta_p (\gamma_m - \gamma)^{1/2}. \]
Once $u$ is found, the density and electric field can immediately be determined using Eq. (73) and Eq. (77a) respectively:

\[ n_e(\tau) = \frac{\beta_p n_0}{\beta_p - u(\tau)}, \]  \hspace{1cm} (85)

\[ E(\tau) = \frac{Y}{\beta_p} = \pm \sqrt{2(\gamma_m - \gamma(\tau))^{1/2}}. \]  \hspace{1cm} (86)

An exact analytical solution of Eqs. (84–2) can be obtained in the limiting case of $\beta_p = 1$, corresponding to a highly underdense plasma.
Numerical solutions – linear Langmuir wave

Integrate Eq. (84), Eq. (2) for $\beta_p = 1, u_m = 0.2$

![Graph showing the numerical solutions for a linear Langmuir wave.](image)

NB: electric field and density 90° out of phase
Numerical solutions – nonlinear Langmuir waves

Parameters: a) \( \beta_p \simeq 1 \) and \( u_m = 0.9 \); b) \( \beta_p = 0.6 \), \( u_m = 0.55 \)

Typical features: i) sawtooth electric field; ii) spiked density; iii) \textit{lengthening} of the oscillation period by factor \( \gamma \).
Wavebreaking

- Definition: Wave breaks when the fluid velocity exceeds the phase velocity.
- If this happens, then some of the electron charge sheets may ‘cross’ each other, and the wave will lose its coherence.
- Analogy: surface water waves reaching shore.
Maximum electric field

Analyse electric field from 1D solution Eq. (2):

\[ E(\tau) = \frac{Y}{\beta_p} = \pm \sqrt{2(\gamma_m - \gamma(\tau))^{1/2}}. \]

Extremum occurs for \( \gamma = 1 \), point in the oscillation when the electrons are momentarily stationary.
Thus at the wave-breaking point \( \gamma_m = \gamma_p \), we have in physical units:

\[ E_{\text{max}} = \frac{mc\omega_p}{e} \sqrt{2(\gamma_p - 1)^{1/2}}. \]  \hspace{1cm} (87)
For *non-relativistic* phase velocities $v_p \ll c$, we have

$$\gamma_p - 1 = (1 - \beta_p^2)^{-1/2} \approx \beta_p^2/2,$$

so that

$$E_{\text{max}} = \frac{m\omega_p v_p}{e},$$

– cold wave-breaking limit (Dawson, 1962).
Maximum field amplitude - fast waves

For relativistic phase velocities, find

\[ E_{\text{max}} \sim \frac{m \omega_p c}{e} \]

Example

\[
\begin{align*}
  m_e &= 9.1 \times 10^{-28} \text{g} \\
  c &= 3 \times 10^{10} \text{cm} \text{s}^{-1} \\
  \omega_p &= 5.6 \times 10^4 (n_e / \text{cm}^{-3})^{1/2} \\
  e &= 4.8 \times 10^{-10} \text{statcoulomb}
\end{align*}
\]

Field:

\[ E_p \sim 4 \times 10^8 \left( \frac{n_e}{10^{18} \text{ cm}^{-3}} \right)^{1/2} \text{ V m}^{-1} \]
Wavebreaking in warm plasmas
nonrelativistic - Coffey, 1971

Thermal effects act to reduce the maximum attainable wave amplitude, because:

1. plasma pressure resists the tendency for the density to explode
2. thermal electrons moving in the direction of the wave may be trapped at a lower wave amplitude than cold particles would be.

First analysed by Coffey (1971) using a so-called ‘waterbag’ model for the electron distribution function, giving:

\[
\frac{eE_{\text{max}}}{m\omega_p v_p} = \left(1 - \frac{\mu}{3} - \frac{8}{3} \mu^{1/4} + 2 \mu^{1/2}\right)^{1/2}, \tag{90}
\]

where \( \mu = 3k_B T_e/mv_p^2 \).
Generalize waterbag model to include relativistic fluid momenta. Equivalent equation to Eq. (83) for the fluid velocity:

\[
\frac{d^2}{d\tau^2} F(u) = \frac{\omega_p^2 \beta_p^2 u}{\beta_p - u},
\]

(91)

where

\[
F(u) = \frac{1 - \beta_p u}{(1 - u^2)^{1/2}} \left[ 1 + \mu \beta_p^2 \frac{1 - u^2}{(\beta_p^2 - u^2)^{1/2}} \right].
\]
Relativistic wavebreaking in warm plasmas – electric field

Maximum electric field in the limit \( \beta_p \approx 1 \pm \sqrt{2} \omega_p \beta_p (\gamma_m - \gamma)^{1/2} \):

\[
\frac{eE_{\text{max}}}{m\omega_p c} = \mu^{-1/4} (\ln 2 \gamma_p^{1/2} \mu^{1/4})^{1/2},
\]

valid for

\[
\gamma_p \gg \frac{1}{2 \mu^{1/2}} \log 2 \mu^{1/4} \gamma_p^{1/2}.
\]
Maximum electric field in warm plasmas

Wavebreaking amplitude of longitudinal plasma oscillations for different phase velocities. Numerical solutions join up with the cold wavebreaking limits (shown for $\gamma_p = 10$ and $\gamma_p = 100$).

\[ eE_{\text{max}}/m_\omega v_p \]

![](image.png)
Summary of Lecture 5

- Plasma can support large amplitude waves
- Longitudinal & transverse components generally coupled
- Longitudinal waves: → spiky electron density; sawtooth E-field
- Max electric field determined by wave breaking limit:

\[ E_{\text{max}} \sim \frac{m\omega_p c}{e} \]