Part III

Interaction with Single Electrons - Plane Wave Orbits
3 Interaction with Single Electrons

Motion of an Electron in an Electromagnetic Plane Wave

Laboratory frame

Average rest frame

Finite pulse duration

Ponderomotive Force

Summary
Single electron motion in EM plane wave

- Electron momentum in electromagnetic wave with fields $\mathbf{E}$ and $\mathbf{B}$ given by Lorentz equation (cgs units):

$$\frac{dp}{dt} = -e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right), \quad (22)$$
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- Energy equation (taking dot product of $\mathbf{v}$ with Eq. (22))

$$\frac{d}{dt} (\gamma mc^2) = -e(\mathbf{v} \cdot \mathbf{E}), \quad (23)$$

where $p = \gamma m\mathbf{v}$, and $\gamma = (1 + p^2/m^2 c^2)^{\frac{1}{2}}$ is the relativistic factor.
Consider elliptically polarized plane-wave $\mathbf{A}(\omega, k)$ travelling in the positive $x$-direction. Vector potential

$$
\mathbf{A} = (0, \delta a_0 \cos \phi, (1 - \delta^2)^{\frac{1}{2}} a_0 \sin \phi),
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where $\phi = \omega t - kx$ is the phase of the wave; $a_0$ is the normalized amplitude ($v_{os}/c$)
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- Polarization parameter \( \delta \):
  - \( \delta = \{\pm 1, 0\} \) \rightarrow \text{linearly polarized wave}
  - \( \delta = \pm 1/\sqrt{2} \) \rightarrow \text{circular wave.}
Normalized (dimensionless) units

\[ t \rightarrow \omega t \]
\[ x \rightarrow kx \]
\[ v \rightarrow v/c \]
\[ p \rightarrow p/mc \]
\[ A \rightarrow eA/mc^2 \]
\[ E \rightarrow eE/m\omega c \]
\[ B \rightarrow eB/m\omega c \]

Equivalent to setting \( \omega = k = c = e = m = 1 \) (cf. atomic units)
Using the relations

\[ E = -\frac{\partial A}{\partial t} \]

and

\[ B = \nabla \times A = (0, -\frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x}) \]

the perpendicular component of momentum Eq. (22) becomes:

\[ \frac{dp_{\perp}}{dt} = \frac{\partial A}{\partial t} + v_x \frac{\partial A}{\partial x}, \]

where \( p_{\perp0} \) is a constant of motion.
Perpendicular (canonical) momentum

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the perpendicular component of momentum Eq. (22) becomes:

\[ \frac{dp_\perp}{dt} = \frac{\partial A}{\partial t} + v_x \frac{\partial A}{\partial x} \]

which after integrating gives *canonical momentum*:

\[ p_\perp - A = p_\perp^0, \quad (25) \]

where \( p_\perp^0 \) is a constant of motion.
Longitudinal momentum

The longitudinal components of Eq. (22) and Eq. (23) yield a pair of equations which can be subtracted from each other thus:

\[
\frac{dp_x}{dt} - \frac{d\gamma}{dt} = -v_y \left( \frac{\partial A_y}{\partial t} + \frac{\partial A_y}{\partial x} \right) - v_z \left( \frac{\partial A_z}{\partial t} + \frac{\partial A_z}{\partial x} \right).
\]

Because the EM wave is a function of \( t - x \) only, the terms on the RHS vanish identically, so we can immediately integrate the RHS to get:

\[
\gamma - p_x = \alpha,
\]

where \( \alpha \) is a constant of motion still to be determined.
General solution

Use identity

\[ \gamma^2 - p_x^2 - p_{\perp}^2 = 1 \]

and choose \( p_{\perp 0} = 0 \) to get relationship between the parallel and perpendicular momenta:

\[ p_x = \frac{1 - \alpha^2 + p_{\perp}^2}{2\alpha}. \]  \hspace{1cm} (26)

– independent of polarization \( \delta \).
Change of variables: wave phase

Need to specify $\alpha$ and integrate Eq. (25) and Eq. (26). This is simplified by changing variables. Noting that

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{p_x}{\gamma} \frac{\partial \phi}{\partial x} = \frac{\alpha}{\gamma},$$

we have

$$p = \gamma \frac{dr}{dt} = \gamma \frac{d\phi}{dt} \frac{dr}{d\phi} = \alpha \frac{dr}{d\phi}. \quad (27)$$
Special cases: (1) laboratory frame

- Lab frame: the electron initially at rest before the EM wave arrives, so that at $t = 0$, $p_x = p_y = 0$ and $\gamma = 1$.
- From Eq. (26) it follows that $\alpha = 1$. 

Special cases: (1) laboratory frame

- Lab frame: the electron initially at rest before the EM wave arrives, so that at $t = 0$, $p_x = p_y = 0$ and $\gamma = 1$.
- From Eq. (26) it follows that $\alpha = 1$. This leads to the following expression for the momenta in the lab frame:

\[
\begin{align*}
  p_x &= \frac{a_0^2}{4} \left[ 1 + (2\delta^2 - 1) \cos 2\phi \right], \\
  p_y &= \delta a_0 \cos \phi, \\
  p_z &= (1 - \delta^2)^{1/2} a_0 \sin \phi.
\end{align*}
\]
Laboratory frame: solution

With Eq. (27) we can integrate expressions Eqs. (28a–28c) to obtain the lab-frame orbits valid for arbitrary polarization $\delta$:

$$
\begin{align*}
  x &= \frac{1}{4} a_0^2 \left[ \phi + \frac{2\delta^2 - 1}{2} \sin 2\phi \right], \\
  y &= \delta a_0 \sin \phi, \\
  z &= -(1 - \delta^2)^{1/2} a_0 \cos \phi.
\end{align*}
$$

(29)

NB: solution is self-similar in the variables \((x/a_0^2, y/a_0)\)
Laboratory frame: linearly polarized wave ($\delta = 1$)

Figure: For a 1 $\mu$m laser wavelength, the pump strengths $a_0$ correspond roughly to intensities of $10^{17}$, $10^{18}$ and $10^{19}$ Wcm$^{-2}$ respectively.
Laboratory frame: drift velocity

- Longitudinal motion has a *secular* component which will grow in time or with propagation distance.
- Electron starts to *drift* with an average momentum

\[ p_D \equiv \bar{p}_x = \frac{a_0^2}{4} \]

corresponding to a velocity:

\[ \frac{v_D}{c} = \bar{v}_x = \frac{\bar{p}_x}{\bar{\gamma}} = \frac{a_0^2}{4 + a_0^2}, \tag{30} \]

where the overscore denotes averaging over the rapidly varying EM phase \( \phi \).
Laboratory frame: circularly polarized light $(\delta = \pm 1/\sqrt{2})$

- Longitudinal oscillating component at $2\phi$ vanishes identically.
- Transverse motion is *circular* with radius $a_0$ and momentum $p_\perp = a_0/\sqrt{2}$.
- Combines with the linear drift in Eq. (30) to give a *helical* orbit with pitch angle
  \[ \theta_p = p_\perp / p_D = \sqrt{8} a_0^{-1}. \] (31)
Special cases: (2) Average rest frame

- Require drift velocity to vanish: $\bar{p}_x = 0$ in Eq. (26), then:

$$1 + \bar{A}^2 - \alpha^2 = 0.$$ 

- Average over a laser cycle to remove rapidly varying terms, and noting that $\cos^2 \phi = 1/2$ gives:

$$\alpha = \left(1 + \frac{a_0^2}{2}\right)^{1/2} \equiv \gamma_0. \quad (32)$$
Plugging $\alpha$ from Eq. (32) back into Eq. (26) gives the momenta:

\[ p_x = (2\delta^2 - 1) \frac{a_0^2}{4\gamma_0} \cos 2\phi, \]
\[ p_y = \delta a_0 \cos \phi, \]  
\[ p_z = (1 - \delta^2)^{\frac{1}{2}} a_0 \sin \phi. \]  

(33)

Noting that in this case, $p = \gamma_0 \frac{d\mathbf{r}}{d\phi}$, we can integrate again to get the orbits:

\[ x = \left( \delta^2 - \frac{1}{2} \right) q^2 \sin 2\phi, \]

\[ r_\perp = 2(\delta q \sin \phi, -(1 - \delta^2)^{\frac{1}{2}} q \cos \phi), \]  

(34)

where $q = a_0/2\gamma_0$. 

Rest frame: solution
Rest frame: linearly polarized wave ($\delta = 1$)

- Eliminate $\phi$ for linear polarization ($\delta = 1$), to obtain *figure-of-eight* orbit.

$$16x^2 = y^2(4q^2 - y^2).$$ (35)
Rest frame: limits

- Extreme intensity limit, $a_0 \gg 1$:
  
  $$\gamma_0 \to a_0 / \sqrt{2}$$
  $$q \to 1 / \sqrt{2}$$

- ‘fat-8’ orbit with
  
  $$\begin{align*}
  |x|_{\text{max}} &= 1/4, \\
  |y|_{\text{max}} &= \sqrt{2}.
  \end{align*}$$

- Circularly polarized light: $\delta = \pm 1/\sqrt{2}$; $p_x = 0$
  
  $\Rightarrow$ circle with radius
  
  $$\frac{a_0}{\sqrt{2}\gamma_0}.$$
Finite pulse duration

\[ A(x, t) = a_0 f(t) \cos \phi, \]  

(36)

Adiabatic approximation: \( df/dt \ll \omega f \).

Example

\[ f(t) = \sin(\pi t/2t_L), \text{ with } \omega t_L = 600/\pi. \]
Ponderomotive force

- Single electron oscillating slightly off-centre of focused laser beam:

  \[ f_p \]

  \[ \mathbf{E}(r) \]

  \[ e^{-i \mathbf{E}_y(r)} \]

- After 1st quarter-cycle, sees lower field
- Doesn’t quite return to initial position

\[ \Rightarrow \] Accelerated away from axis
Ponderomotive force: non-relativistic

In the limit $v/c \ll 1$, the equation of motion (22) for the electron becomes:

$$\frac{\partial v_y}{\partial t} = -\frac{e}{m} E_y(r).$$

(37)

Taylor expanding electric field:

$$E_y(r) \simeq E_0(y) \cos \phi + y \frac{\partial E_0(y)}{\partial y} \cos \phi + \ldots,$$

where $\phi = \omega t - kx$ as before.

To lowest order, we therefore have

$$v_y^{(1)} = -v_{os} \sin \phi; \quad y^{(1)} = \frac{v_{os}}{\omega} \cos \phi,$$

where $v_{os} = eE_L/m\omega$. 
Ponderomotive force: non-relativistic (contd.)

Substituting back into Eq. (37) gives

$$\frac{\partial v_y^{(2)}}{\partial t} = -\frac{e^2}{m^2 \omega^2} E_0 \frac{\partial E_0(y)}{\partial y} \cos^2 \phi.$$ 

Multiplying by $m$ and taking the cycle-average yields the ponderomotive force on the electron:

$$f_p \equiv m \frac{\partial v_y^{(2)}}{\partial t} = -\frac{e^2}{4m\omega^2} \frac{\partial E_0^2}{\partial y}.$$ (38)
Lorentz equation (22) in terms of the vector potential $\mathbf{A}$:

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{p} = \frac{e}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{e}{c} \mathbf{v} \times \nabla \times \mathbf{A}. \quad (39)$$

Separate the timescales of the electron motion into slow and fast components $\mathbf{p} = \mathbf{p}^s + \mathbf{p}^f$ and average over a laser cycle,

$$\mathbf{f}_p = \frac{d \mathbf{p}^s}{dt} = -mc^2\nabla \gamma, \quad (40)$$

where $\gamma = \left(1 + \frac{p_s^2}{m^2c^2} + \frac{1}{2}a_0^2\right)^{1/2}$. 
Summary: electron motion in plane wave
Linear polarisation

Solution in terms of wave phase
\( \phi = \omega t - kx \):

\[
x = \frac{1}{4} a_0^2 \left( \phi + \frac{1}{2} \sin 2\phi \right),
\]
\[
y = a_0 \sin \phi
\]

Drift velocity:
\[
\frac{v_D}{c} = \frac{a_0^2}{4 + a_0^2}
\]