Geometric phases in strongly interacting, driven open quantum systems

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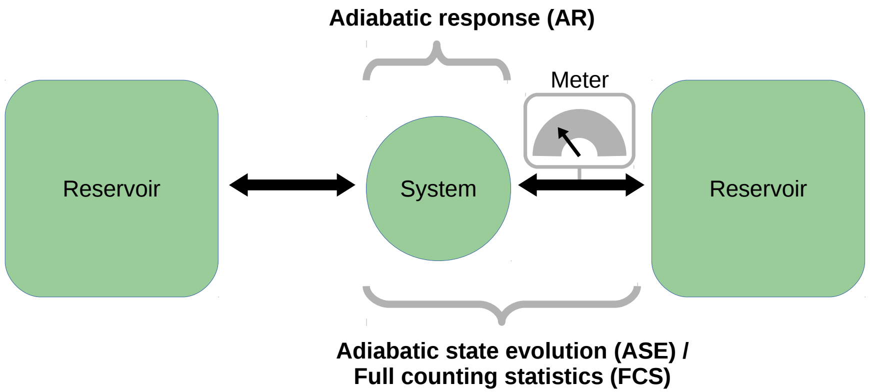
Following the book chapter:

I. **Geometry, topology and physics**
→ **Open-systems**



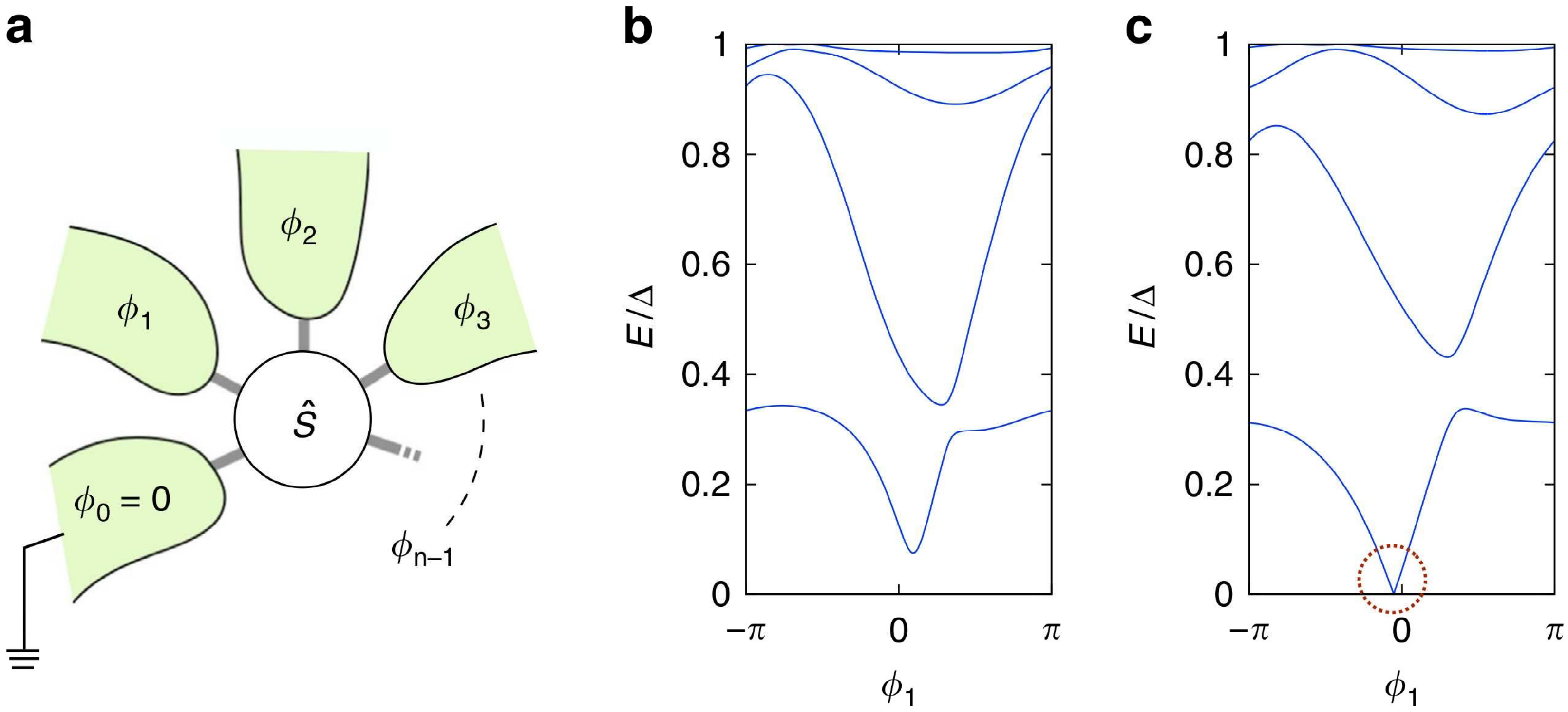
II. **Adiabatic response in transport**
→ **Simple, transparent**

III. **Full counting statistics**
→ **General, similar to closed systems**

IV. **Adiabatic state evolution**
→ **Combine I + II + III to resolve puzzles**

I Geometry, topology and physics

Circuits with n terminals $\sim n-1$ dimensional **topological** material Riwar et. al [17]



⇒ Need clear general view of **underlying** (differential) **geometry** of open systems

- Dissipation, nonequilibrium = **essential** ingredient
- Circuits $\sim$ **space** of **charge** $\hat{N}$ + conjugate **phase** $\hat{\chi}$ [16], [13]

$$[\hat{N}, \hat{\chi}] \;=\; i$$

1 Physics and geometry

Geometric open-system approaches **including measurements**

- (seem to) have **no direct link** with **closed-system** Berry-Simon approach
- (seem to) say **physically opposite things**...

AR	**FCS**
Nonadiabatic	Adiabatic
Stationary	Nonstationary
Landsberg phase	Berry-Simon phase

?

- produce **same results** Sinitsyn [24], Nakajima et. al [14], Plücker et. al [30]

$$\boxed{\Delta N^{\text{AR}} = \Delta N^{\text{FCS}}} \quad !$$

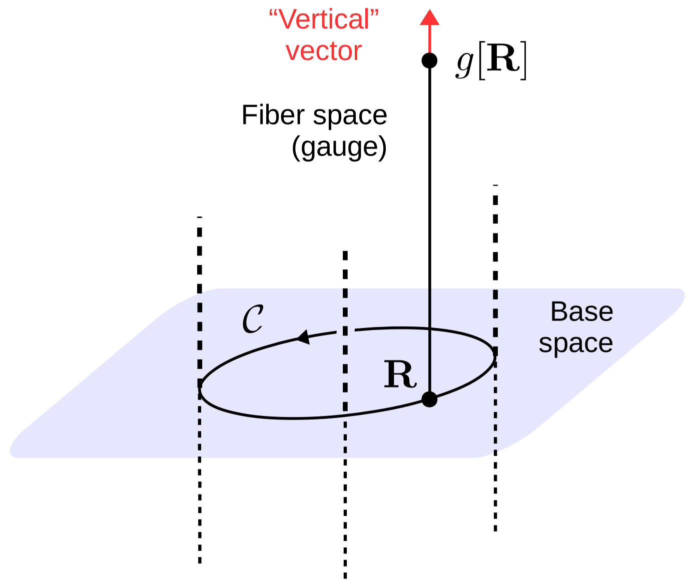
•• Comparing geometric approaches

1. **In which physical space** are we solving a problem **?**

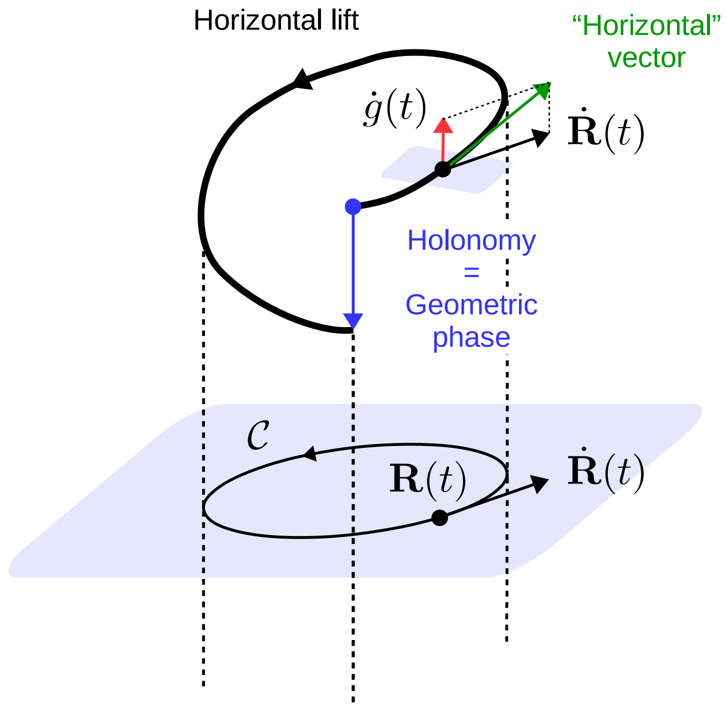
$$\text{Fiber bundle} \;=\; \text{base} \times \text{fibers} \quad (\textit{naive})$$

Fibers **define** "vertical" but

"Horizontal" is **not defined** in such a space



2. **Restrictions of physics** expressible as **geometry ?**
   - Connection A = **definition** of "horizontal"
   - Curvature B = "horizontal lifts" tend to break



Topology of fiber bundle ("*twisted product*")

$$\int \mathrm{d}R \,\text{polynomials}(B) \;\in\; \mathbb{Z} \qquad \text{NOT DISCUSSED}$$

2 Adiabatic state evolution of closed quantum systems

$$\frac{d}{dt}|\psi\rangle \;=\; -i\,\hat{H}\,|\psi\rangle$$

with parametric driving $\hat{H}(t)=\hat{H}[R(t)]$

$$|\psi(t)\rangle \;=\; \textbf{?}$$

●● Adiabatic approximation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle \;=\; -i\,\hat{H}\,|\psi\rangle$$

1. **Parametric stationary solutions** $|h_n[R]\,\rangle$

$$\hat{H}\,|h_n\rangle \;=\; h_n\,|h_n\rangle$$

2. **Gauge freedom** (unitary, compact)

$$|h_n\rangle \;\to\; g_n\,|h_n\rangle, \qquad |g_n| \;=\; 1, \qquad g_n[R]$$ **continuous** and smooth

3. **Eigenspace decoupling** for nonstationary state: with $h_n(t) = h_n[R(t)]$

$$|\psi(t)\rangle \;=\; \sum_n c_n(t)\,|h_n(t)\rangle, \qquad c_n(0) \;=\; \langle h_n(0)|\;\psi(0)\,\rangle$$

when projecting with $\langle h_n|\bullet$

$$\langle h_n|\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle \;\approx\; \frac{\mathrm{d}}{\mathrm{d}t}c_n\,\langle h_n|h_n\rangle + c_n\,\langle h_n|\frac{\mathrm{d}}{\mathrm{d}t}|h_n\rangle \;\stackrel{!}{=}\; -i\,h_n\cdot c_n \;=\; -i\,\langle h_n|\hat{H}\,|\psi\rangle$$

•• Adiabatic pure quantum state

$$|\psi(T)\rangle \approx \sum_n e^{\int_0^T \mathrm{d}t\left(-i h_n - \langle h_n | \frac{\mathrm{d}}{\mathrm{d}t} | h_n\rangle\right)} \cdot c_n(0) \cdot | h_n[R(T)] \rangle$$
$$= \sum_n e^{\int_0^T \mathrm{d}t(-i h_n) - \int_{\mathcal{C}} \mathrm{d}R\, A_n[R]} \cdot c_n(0) \cdot | h_n[R(0)] \rangle$$

- **Eigenvalues** $\Rightarrow$ nongeometric factor $e^{\int_0^T \mathrm{d}t(-i h_n)}$
- **Eigenvectors** $\Rightarrow$ geometric factor $e^{-\int_{\mathcal{C}} \mathrm{d}R\, A_n[R]}$ with **Berry-Simon connection**

$$A_n[R] := \langle h_n | \nabla_R | h_n \rangle$$

Because we consider **quantum state** (**not** true for other objects!)

($math$) Eigenspace decoupling = adiabatic approximation ($physics$)

$$\langle h_n|\frac{\mathrm{d}}{\mathrm{d}t}|h_n\rangle \;=\; \langle h_n|\dot{R}\cdot\nabla_R|h_n\rangle \;=\; \dot{R}(t)\cdot A_n[R(t)]$$

$$\int_0^T \mathrm{d}t\,\dot{R}(t)\,A_n[R(t)] \;=\; \int_{\mathcal{C}}\mathrm{d}R\,A_n[R]$$

•• Gauge freedom connection – gauge invariance solution

Under change of gauge of each eigenvector ($g_n[R]$ continuous, smooth)

$$|h_n\rangle \;\rightarrow\; g_n|h_n\rangle \;=\; e^{i\varphi_n}|h_n\rangle$$

- Connection changes[1]

$$A_n \;\rightarrow\; \boxed{A_{n,g} \;=\; A_n + \frac{1}{g_n}\nabla_R\, g_n} \;=\; A_n + \nabla_R\,(i\varphi_n)$$

- but geometric factor **invariant**

$$e^{-\int_{\mathcal{C}} \mathrm{d}R\, A_n[R]} \;=\; e^{-\int_{\mathcal{C}} \mathrm{d}R\, A_{n,g}[R]}$$

Not artifact of coordinate choice !

1. $A_{n,g} \;=\; \Big(\langle h_n|\frac{1}{g_n}\Big)\nabla_R\Big(g_n|h_n\rangle\Big) \;=\; \frac{1}{g_n}g_n\cdot\langle h_n|\nabla_R|h_n\rangle + \frac{1}{g_n}\nabla_R\, g_n\cdot\langle h_n|h_n\rangle \;=\; A_n + \frac{1}{g_n}\nabla_R\, g_n$

What is a “geometric effect” ? (systematically)

•• Guiding questions

1. **Fiber bundle?** Total physical space

$$(R \ , \ g_n \ ) \ = \ \left(\begin{matrix} \text{driving} \\ \text{parameter} \end{matrix} \ , \ \begin{matrix} \textbf{eigenstate } |h_n\rangle \\ \text{normalization} \end{matrix} \right)$$

2. **Connection?** **Broken** physical solution curve

$$g_n(T) \ = \ e^{-\int_0^T \mathrm{d}t \, \dot{R} \cdot A_n} \cdot g_n(0) \ \neq \ 1 \cdot g_n(0)$$

= **unavoidable phase** due to cyclic driving

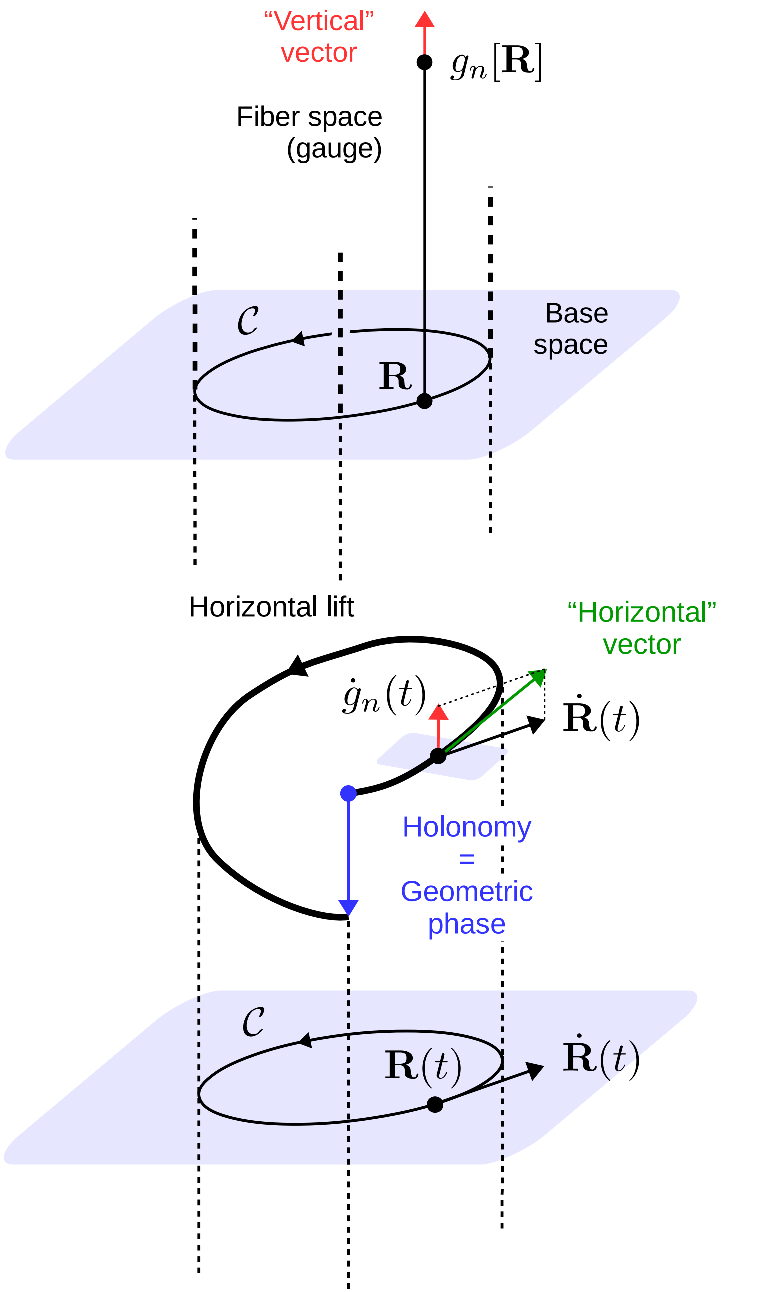
Try to "gauge away" A_n from solution

$$A_{n,g} \cdot \dot{R} \ = \ A_n \cdot \dot{R} + \frac{1}{g_n} \cdot \dot{g}_n \ \overset{?}{=} \ 0$$

= maintain **constant phase** for $|h_n\rangle$

$(\ \dot{R} \ , \ \dot{g}_n \ )$ defines "**horizontal**" vector

"**Horizontal lift**" / "**parallel transport**" of vector **breaks closed** base curve $\mathcal{C}$



●● Curvature

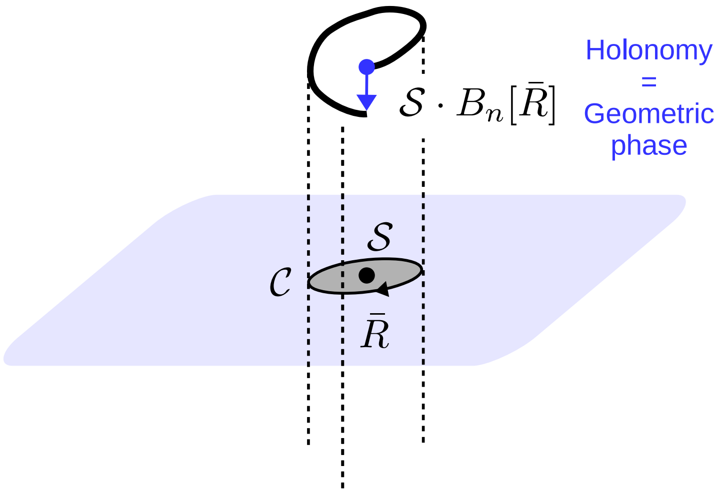
2. **Curvature?**
   = "measure of **breakage** of horizontal lift curves"

$$B_n \; = \; \nabla_R \times A_n \; = \; \langle \nabla_R h_n | \times | \nabla_R h_n \rangle$$

For infinitesimal curve $\mathcal{C}$ around $\bar{R}$ enclosing area $\mathcal{S} \to 0$

$$g_n(T) \; = \; e^{-\int_{\mathcal{C}} \mathrm{d}R\, A_n[R]} \cdot g_n(0)$$

$$\overset{\text{Stokes}}{\approx} \; \left(1 - \mathcal{S} \cdot B_n[\bar{R}]\right) \cdot g_n(0)$$



What is kept constant ? (Phase $\langle h_n|\frac{\mathrm{d}}{\mathrm{d}t}|h_n\rangle$ of each state in superposition)

3 Adiabatic state evolution of open quantum systems

•• Open systems (1)

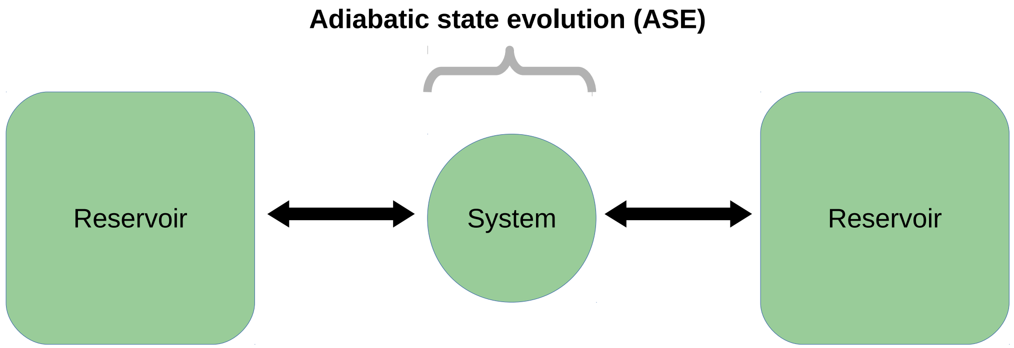
- Open system in **contact** (V)

$$\hat{H}^{\mathrm{tot}} = \underset{\uparrow}{\hat{H}} + \hat{H}^{\mathrm{R}} + \hat{V}$$

interactions

with noninteracting equilibrium reservoirs

$$\hat{\rho}^{\mathrm{R}} = \prod_{r=L,R} \frac{1}{Z^r} e^{-(\hat{H}^r - \mu^r \hat{N}^r)/T^r}$$



- Schrödinger dynamics (Liouville - von Neumann)

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}^{\mathrm{tot}} = -i\left[\hat{H}^{\mathrm{tot}}, \hat{\rho}^{\mathrm{tot}}\right]$$

- **Reduced density operator** incorporates reservoirs

$$\hat{\rho} := \underset{\mathrm{R}}{\mathrm{tr}}\,\hat{\rho}^{\mathrm{tot}} \quad \text{mixed quantum state}$$

•• Open systems (2)

- Hilbert-Schmidt / Liouville **supervector**

$$\hat{\rho} \;\cong\; \begin{bmatrix} \ddots & & \\ & \rho_{ij} & \\ & & \ddots \end{bmatrix} \;\longrightarrow\; |\rho) \;\cong\; \begin{bmatrix} \vdots \\ \vdots \\ \rho_{ij} \\ \vdots \\ \vdots \end{bmatrix}$$

- **Trace** ⇒ scalar product for 2 operators A and B

$$(A|\bullet \;=\; \mathrm{tr}\hat{A}^\dagger\bullet \quad\longleftarrow\quad (A|B) \;=\; \mathrm{tr}\hat{A}^\dagger\hat{B} \;=\; \sum_{ij} A^*_{ij}B_{ij} \quad\longrightarrow\quad |B) \;=\; \hat{B}$$

$$\boxed{(\mathbb{1}|\bullet \;=\; \mathrm{tr}\,\mathbb{1}\bullet \;=\; \mathrm{tr}\bullet}$$

- **Evolution equation**: ignoring memory effects:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho) \;=\; -i\,L|\rho) + W|\rho)$$

  - Liouvillian superoperator $L\,\hat{\rho} = [\hat{H}\,,\,\hat{\rho}]$ ⇒ Hamiltonian dynamics
  - Memory kernel superoperator W ⇒ nonequilibrium, dissipation

●● Adiabatic approximation (1)

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho) = W|\rho)$$

with parametric driving $W(t) = W[R(t)]$ Sarandy, Lidar [19],[20]

$$|\rho(t)) = \textbf{?}$$

$W =$ **not** (skew) hermitian matrix

- **Non**orthogonal eigenvectors = different left / right eigenvectors [2]
- **Zero** eigenvectors

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\,\hat{\rho}(t) = \mathrm{tr}\,W\hat{\rho}(t) = (\mathbb{1}|W|\rho(t)) = 0 \quad \Rightarrow \quad \begin{cases} \mathrm{tr}\,\hat{\rho}(t) = 1 & \text{probability conservation} \\ \lim_{t\to\infty}\hat{\rho}(t) \text{ exists} & \text{stationary state (unique)} \end{cases}$$

2. There is a nontrivial duality for general open fermion systems [22], [28] in wide-band limit: the left eigenvectors for a **repulsive** system ∼ right ones for dual **attractive** system.

•• Adiabatic approximation (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho) = W|\rho)$$

1. **Parametric solutions**:
   Amplitude covectors $\hat{\bar{w}}_n$ ≠ decay modes $\hat{w}_n$ + complex eigenvalue w_n ($\mathrm{Re}\, w_n \leqslant 0$)

$$(\bar{w}_n|W = (\bar{w}_n|w_n, \qquad W|w_n) = w_n|w_n)$$

   Zero eigenvalue:

$$(\bar{w}_0| = (\mathbb{1}| = \mathrm{tr}\,\bullet \quad \Leftarrow \quad (\bar{w}_0|W = 0, \qquad W|w_0) = 0 \quad \Rightarrow \lim_{t\to\infty} |\rho(t)) = |w_0)$$

2. **Gauge freedom** preserving $(\bar{w}_n|w_{n'}) = \delta_{nn'}$ (**non**unitary, **non**compact)

$$(\bar{w}_n| \to (\bar{w}_n| \cdot \frac{1}{g_n}, \qquad |w_n) \to g_n \cdot |w_n), \qquad g_n \neq 0$$

   Zero eigenvectors:

$$g_0 = 1 \text{ probability conservation}$$

3. **Eigenspace decoupling** for nonstationary state: with $w_n(t) := w_n[R(t)]$

$$c_n(0) = (\bar{w}_n|\rho(0)) = \mathrm{tr}\,\hat{\bar{w}}_n^\dagger \rho(0), \qquad |\rho(t)) = 1 \cdot |w_0(t)) + \sum_{n\geqslant 1} c_n(t)\,|w_n(t))$$

- **Fixed parameters** R:
  stationary (surviving) + **nonstationary** (decaying) part

$$|\rho(t)) \;=\; e^{W[R]\,t} \cdot |\rho(0)) \;=\; 1 \cdot |w_0[R]\,) + \sum_{n \geqslant 1} e^{w_n[R]\,t} \cdot c_n(0) \cdot |w_n[R]\,) \;\xrightarrow{t \to \infty}\; \underset{\substack{\uparrow \\ \text{constant !}}}{|w_0[R])}$$

Eigenvectors:

$$W|w_n) \;=\; w_n|w_n)$$

- **Driven parameters** $R(t)$: after transients
  adiabatic steady state = **parametrically stationary** + **nonstationary** part

$$|\rho(t)) \;=\; \ldots. \;\xrightarrow{t \to \infty}\; 1 \cdot |w_0[R(t)]\,) + \sum_{n \geqslant 1} c_n(t) \cdot |w_n[R(t)]\,)$$

Amplitude (super)covectors $\hat{\bar{w}}_n \neq \hat{w}_n$ Decay **mode** (super)vectors

- Right eigenvectors = decay **modes**

$$W|w_n) = \begin{bmatrix} & \vdots & \\ \cdots & W_{ij,i'j'} & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ (w_n)_{i'j'} \\ \vdots \end{bmatrix} \leftarrow \textbf{not } (\bar{w}_n)_{ij}$$

- Left eigenvectors = decay **amplitude**

$$(\bar{w}_n|W = \begin{bmatrix} \cdots & (\bar{w}_n)^*_{ij} & \cdots \end{bmatrix} \begin{bmatrix} & \vdots & \\ \cdots & W_{ij,i'j'} & \cdots \\ & \vdots & \end{bmatrix}$$

$\uparrow$ **not** $(w_n)^*_{ij}$

●● Adiabatic mixed quantum state

$$
\begin{aligned}
|\rho(T)) &\approx 1 \cdot |w_0[R(T)]) + \sum_{n \geqslant 1} e^{\int_0^T \mathrm{d}t \left(w_n - (\bar{w}_n | \frac{\mathrm{d}}{\mathrm{d}t} | w_n) \right)} \cdot c_n(0) \cdot |w_n[R(T)]) \\
&= 1 \cdot |w_0[R(0)]) + \sum_{n \geqslant 1} e^{\int_0^T \mathrm{d}t\, w_n - \int_C \mathrm{d}R\, A_n[R]} \cdot c_n(0) \cdot |w_n[R(0)])
\end{aligned}
$$

parametric **stationary** | parametric **nonstationary**

- **Eigenvalues** $\Rightarrow$ nongeometric factor $e^{\int_0^T \mathrm{d}t\, w_n}$ $\Rightarrow$ **decay** + oscillation

- **Eigenvectors** $\Rightarrow$ geometric factor $e^{-\int_C \mathrm{d}R\, A_n[R]}$ with **Berry-Simon type connection**

$$
A_n[R] \;:=\; (\bar{w}_n | \nabla_R | w_n)
$$

In arbitrary gauge:

$$
A_n \;\rightarrow\; A_{n,g} \;=\; A_n + \frac{1}{g_n} \nabla_R\, g_n
$$

(*math*) Eigenspace decoupling = adiabatic approximation (*physics*)

●● End of story ?

Driving slower than *rate* of exchange with invironment

$$\dot{R} \ll \Gamma$$

$$\Rightarrow \quad |\rho(T)) = \underset{\uparrow}{1} \cdot |w_0[R(0)])$$

No mixed steady-state **Berry-Simon phase** of driven open system !

$\Rightarrow$ Fundamental reason: probability conservation ! Plücker et al. [30]

Assumptions:

- No memory (Born-Markov)
- Frozen parameter approximation
- Unique stationary state

•• Where is the Berry-Simon phase ?

$$|\rho(T)) \;=\; 1\cdot|w_0[R(0)]) + \sum_{n\geqslant 1} e^{\int_0^T \mathrm{d}t\, w_n - \int_{\mathcal{C}} \mathrm{d}R\, A_n[R]} \cdot c_n(0)\cdot|w_n[R(0)])$$

$$\uparrow$$

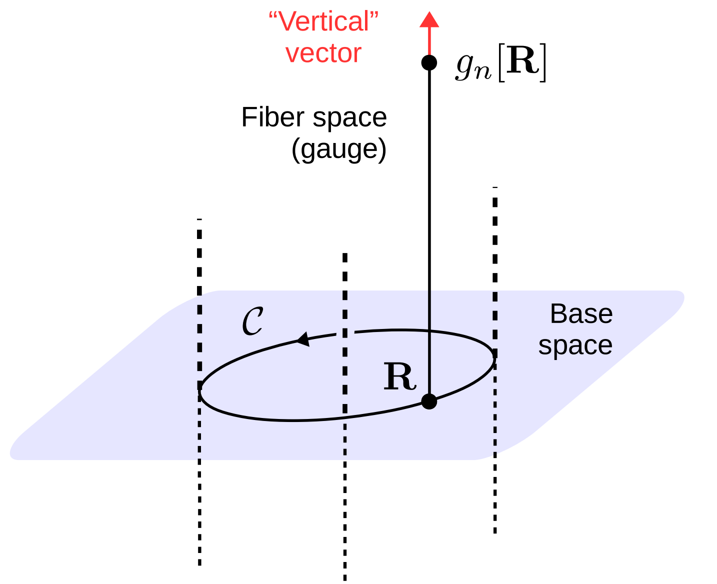
Berry-Simon phase located in parametrically **nonstationary** part of mixed **state** !

Berry-Simon phase will be "revived" tomorrow in Part IV → p. 61

●● Guiding questions

1. **Fiber bundle?** Total physical space

$$(\,R\,,\ g_n\,)\ =\ \left(\begin{matrix}\text{driving}\\ \text{parameter}\end{matrix}\,,\ \begin{matrix}\textbf{nonstationary mode}\\ |w_n\rangle\ \text{normalization}\end{matrix}\right)$$



2. **Connection?** **Broken** physical solution curve

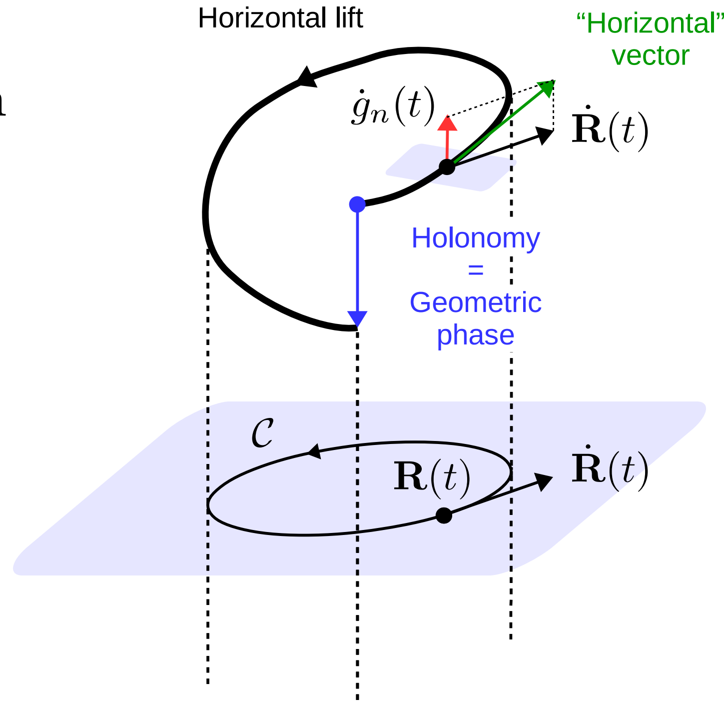
$$g_n(T)\ =\ e^{-\int_0^T \mathrm{d}t\,\dot R\cdot A_n}\cdot g_n(0)\ \neq\ 1\cdot g_n(0)$$

= **unavoidable extra decay** due to cyclic driving

“Horizontal lift” : **Try to** “gauge away” A_n from solution

$$\dot R\cdot A_{n,g}[R]\ =\ A_n\cdot\dot R+\frac{1}{g_n}\cdot\dot g_n\ \overset{?}{=}\ 0$$

= maintain **zero extra decay**



Curvature? Nonzero value of

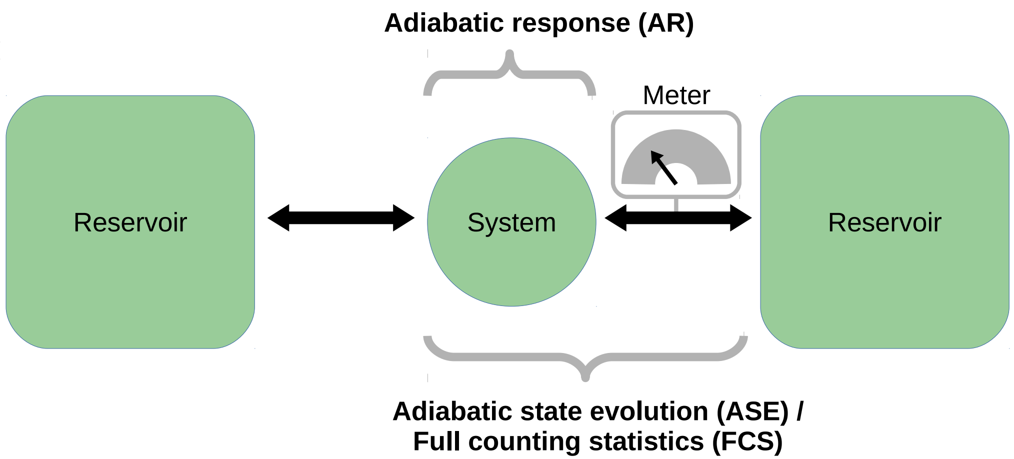
$$B_n\ =\ \nabla\times A_n\ =\ (\nabla_R\,\bar w_n|\times|\nabla_R\,w_n)$$

measures **failure to eliminate extra decay**

$$\dot{R} \cdot A_{n,g}[R] \;=\; \dot{R} \cdot A_n + \frac{1}{g_n} \dot{R} \cdot \nabla_R \, g_n \;=\; \dot{R} \cdot A_n + \frac{1}{g_n} \, \dot{g}_n$$

•• Probing an open system – charge measurements

Charge $\hat{N}$ measured in electrode r **outside** system:



$$\Delta N = \langle \hat{N} \rangle(T) - \langle \hat{N} \rangle(0) \qquad \text{FCS} \to \text{Part III}$$

$$= \int_0^T \mathrm{d}t \, \langle \hat{I}_N \rangle(t) \qquad \text{AR} \to \text{Part II}$$

(similar: spin, energy / heat, etc.)

Gauge freedom emerges in **periodically driven transport**

$$\langle \hat{N} \rangle(t) \to \langle \hat{N} \rangle(t) + g[R(t)] \qquad \text{FCS}$$

$$\langle \hat{I}_N \rangle \to \langle \hat{I}_N \rangle + \frac{\mathrm{d}}{\mathrm{d}t} g[R(t)] \qquad \text{AR}$$

(Rough spoiler):	Transported **charge** = geometric **phase** Transport **current** = geometric **connection**

II Adiabatic-response in transport

$$\Delta N = \int_0^T \mathrm{d}t \, \langle \hat{I}_N \rangle \Big(R(t), \dot{R}(t), \dots \Big) \approx \int_0^T \mathrm{d}t \, I_N[R(t)] + \int_0^T \mathrm{d}t \, \dot{R}(t) \cdot A[R(t)]$$

$$= \int_0^T \mathrm{d}t \, I_N[R(t)] + \int_{\mathcal{C}} \mathrm{d}R \, A[R]$$

↑

Pumping: even there if $I_N[R] = 0$!

$$I_N[R] := \langle \hat{I}_N \rangle \Big|_{\dot{R}=0,\dots}$$

adiabatic current: "frozen parameters"

$$A[R] := \frac{\delta \langle \hat{I}_N \rangle}{\delta \dot{R}} \Bigg|_{\dot{R}=0,\dots}$$

adiabatic response = **nonadiabatic** current

Pumping current = **geometric** *because* it is **nonadiabatic**

↓

$$\delta \langle \hat{I}_N \rangle (t) := \dot{R}(t) \cdot A[R(t)]$$

It is a correction relative to parametric **stationary** state

4 Adiabatic-response approach to pumping

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}|\rho) = W|\rho),} \qquad \boxed{\langle \hat{I}_N \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{N} \rangle = (\mathbb{1}|W_{I_N}|\rho)} = \mathrm{tr}\,(W_{I_N}\rho)$$

1. Expand around **adiabatic** stationary state = parametric zero-mode

$$W[R] \cdot |w_0[R]) = 0$$

2. Solve **nonadiabatic** correction $\propto \dot{R} \ll \Gamma \sim W$ (relaxation rate)

$$|\rho) \approx |w_0) + \frac{1}{W} \cdot \frac{\mathrm{d}}{\mathrm{d}t}|w_0) = |w_0) + \frac{1}{W} \cdot \dot{R}\,\nabla_R\,|w_0)$$

Problem solved:

$$\boxed{\Delta N = \int_0^T \mathrm{d}t\, I_N[R(t)] + \int_{\mathcal{C}} \mathrm{d}R\, A[R]}$$

$$I_N[R] := (\mathbb{1}|W_{I_N}|w_0) \qquad A[R] := (\mathbb{1}|W_{I_N}\frac{1}{W}\nabla_R\,|w_0)$$

Landsberg [10],[11], Berry, Robbins [5], Splettstoesser et. al [27], Avron et. al [3],[2],[1]

1. Expand around adiabatic = parametric stationary state

$$|\rho(t)) \;\approx\; |w_0[R(t)]) + |\rho^n(t))$$

$$\uparrow \quad \propto \dot{R}$$

$\Rightarrow$ linearize in driving velocity $\dot{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}|w_0[R(t)]) + \frac{\mathrm{d}}{\mathrm{d}t}|\rho^n(t)) \;\stackrel{!}{=}\; W[R(t)]\,|w_0[R(t)]\,) + W|\rho^n(t)) \;=\; W|\rho(t))$$

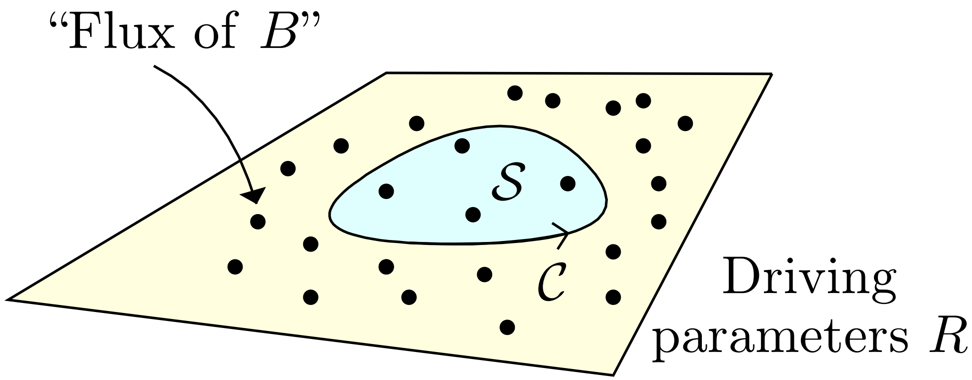
$\uparrow$ $=0$ stationary!

2. Solve for nonadiabatic correction

$$|\rho^n) \;\approx\; \frac{1}{W}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|w_0)$$

•• Pumping curvature (1)

$$\Delta N = \int_0^T \mathrm{d}t\, I_N[R(t)] + \int_{\mathcal{C}} \mathrm{d}R\, A$$
$$\stackrel{\text{Stokes}}{=} \int_0^T \mathrm{d}t\, I_N[R(t)] + \int_{\mathcal{S}} \mathrm{d}S B,$$



- **Pumping curvature**

$$B = \nabla_R \times A$$

$=$ pumped charge per unit area as $\mathcal{S} \to 0$

- **Geometric protection**:
Small driving amplitude $\Rightarrow$ small cycle $\mathcal{C}$ around working point $\bar{R}$

$$\Delta N \approx T \cdot I_N[\bar{R}] + \mathcal{S} \cdot B(\bar{R})$$

$\Rightarrow$ Pumped charge depends only on geometric area $\mathcal{S}$ of driving curve (not it's *shape*)

•• Pumping curvature (2)

Pumping connection / curvature:

$$A = (\Phi_N|\nabla_R|w_0), \qquad B = (\nabla_R \Phi_N| \times |\nabla_R w_0)$$

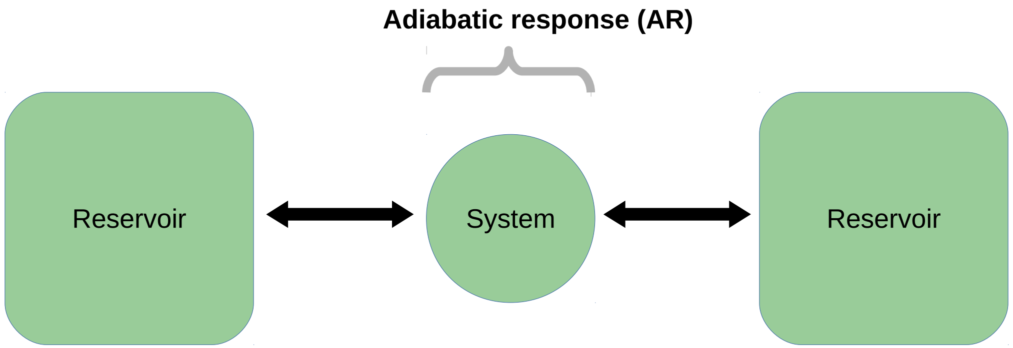
- $|\nabla_R w_0) =$ sensitivity to parameter change of parametric stationary **state**
- $(\nabla_R \Phi_N| =$ sensitivity of **response** (super) **covector** for **charge** Calvo et. al [7]

$$(\Phi_N| = (\mathbb{1}|W_{I_N} \cdot \frac{1}{W}$$ ∼ current × relaxation time
∼ charge transported during relaxation

How can *observable* pick up geometric phase?

5 Landsberg's geometric phase

Charge N measured
in reservoir r **outside** system



Gauge freedom
= **physical recalibration** of charge meter in parametrically **time-dependent** way:

$$\hat{N} \rightarrow \hat{N}_g = \hat{N} + g[R]\,\mathbb{1}$$

- **Pumping current** is **altered** at all times

$$\delta\langle \hat{I}_{N_g}\rangle = \left.\frac{\delta\langle \hat{I}_{N_g}\rangle}{\delta \dot{R}}\right|_{\dot{R}=0} \dot{R} = A_g \cdot \dot{R}, \qquad A_g = A + \nabla_R\, g$$

- but periodically **pumped charge** is **gauge invariant**

$$\Delta N \overset{!}{=} \int_0^T \mathrm{d}t\, I_N[R(t)] + \int_{\mathcal{C}} \mathrm{d}R\, A_g[R] \overset{!}{=} \Delta N_g$$

●● Physical origin of geometric pumping phase

*"You can **time-dependently** mess around all you like with the scale of a charge **meter**, as long as you **periodically** return it to **measure** the correct pumped charge."*

•• Gauge-covariant transport theory (Skip)

Covariant transport equation: [30]

$$\langle \hat{I}_{N_g} \rangle \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{N}_g \rangle \;=\; (\mathbb{1}|W_{I_{N_g}}|\rho)$$

$$\hat{N} \;\rightarrow\; \hat{N}_g = \hat{N} + g\,\mathbb{1} \quad \text{(reservoir)} \qquad\qquad W_{I_N} \;\rightarrow\; W_{I_{N_g}} \;=\; W_{I_N} + \frac{\mathrm{d}}{\mathrm{d}t} g\,\mathbb{I} \quad \text{(system)}$$

Two requirements:

1. Keep explicit time-dependence

$$\hat{I}_N \;=\; i\big[\hat{H}^{\mathrm{tot}}, \hat{N}\big] + \frac{\partial \hat{N}}{\partial t}$$

2. Normal order current with respect to environment

$$\hat{I}_N \;=\; \langle \hat{I}_N \rangle^{\mathrm{res}} + :\hat{I}_N:$$

- $\langle \hat{I}_N \rangle^{\mathrm{res}} =$ "average" = time-local terms → gauge freedom: **exact**
- $:\hat{I}_N: \;=$ "fluctuation" = time-nonlocal terms → memory: **approximations**

•• Guiding questions

1. **Fiber bundle?** Total physical space

$$(R \ , \ g \ ) \ = \ \left(\begin{array}{c} \text{driving} \\ \text{parameter} \end{array} , \begin{array}{c} \hat{N}\textbf{-meter} \\ \textbf{calibration} \end{array} \right)$$

↑ literal gauge !!!

2. **Connection?** **Broken** physical solution curve

$$g(T) - g(0) \ = \ -\int_0^T \mathrm{d}t \, \dot{R} \, A[R] \ = \boxed{ -\int_0^T \mathrm{d}t \, \delta \langle \hat{I}_N \rangle }$$

⇒ **Unavoidable pumped charge** in cyclic dynamics

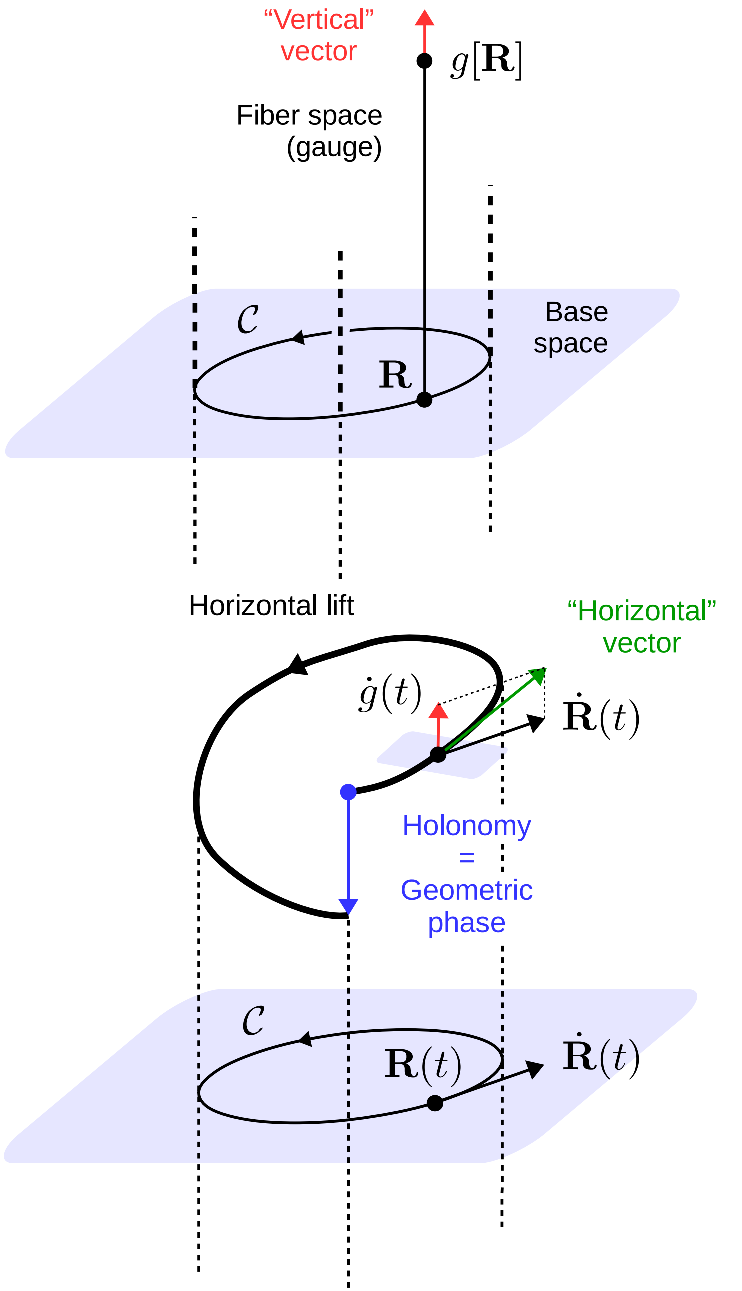
"Horizontal lift" = **Try to** gauge away A from solution:

$$\dot{R} \, A_g \ = \ A\dot{R} + \dot{g} \ = \boxed{ \delta \langle \hat{I}_{N_g} \rangle \ \stackrel{?}{=} \ 0 }$$

= recalibrate meter to show **zero pumping current**

Curvature = pumped charge / driving area

$$\Delta N \ \stackrel{\mathcal{S} \to 0}{\approx} \ T \cdot I_N[\bar{R}] + \mathcal{S} \cdot B(\bar{R})$$



Different physics, but related gauge groups:

- Addition of charge contributions

$$g(T) \;=\; -\int_0^T \mathrm{d}t\, \dot{R}\, A + g(0) \qquad \text{Landsberg}$$

$$\dot{R}\, A_g \;=\; A\dot{R} + \dot{g}$$

- Multiplication of phase factors

$$g_n(T) \;=\; e^{-\int_0^T \mathrm{d}t\, \dot{R}\, A_n} \cdot g_n(0) \qquad \text{Berry-Simon}$$

$$\dot{R} \cdot A_{n,g} \;=\; A_n \cdot \dot{R} + \frac{1}{g_n} \cdot \dot{g}_n$$

$\Rightarrow$ Part III+IV $\rightarrow$ p. 64: Berry-Simon phase of FCS generating operator ($\neq$ state!)

$$\begin{array}{c} \text{Landsberg} \\ \downarrow \\ \text{Berry-Simon} \;\longrightarrow\; e^{i\chi g(T)} \;=\; e^{-i\chi \int_0^T \mathrm{d}t\, \dot{R}\, A} \cdot e^{i\chi g(0)} \end{array}$$

●● Pupmed charge = Landsberg phase

Pumped charge	=	**Landsberg geometric phase** of **observable**
	≠	**Berry-Simon** phase of the **state**

6 Questions

Advantages:

1. **Calculations** simple

$$B \;=\; (\mathbb{1}|\nabla_R\Big(W_{I_N}\cdot\frac{1}{W}\Big)\times\nabla_R\,|w_0)$$

2. **Physics** clear

$$\Delta N \;\overset{\mathcal{S}\to 0}{\approx}\; \ldots + \mathcal{S}\cdot B$$

3. **Geometry** clear:

$$\text{meter calibration} \quad \hat{N} \;\to\; \hat{N}_g \;=\; \hat{N}+g \quad \text{gauge transformation}$$

$$\text{pumping current} \quad \delta\langle\hat{I}_{N_g}\rangle \;=\; A\cdot\dot{R}+\dot{g} \quad \text{geometric connection}$$

.... but **“alien” to closed system** formulations ...

Berry-Simon approach **analogous** to **closed-systems** ?

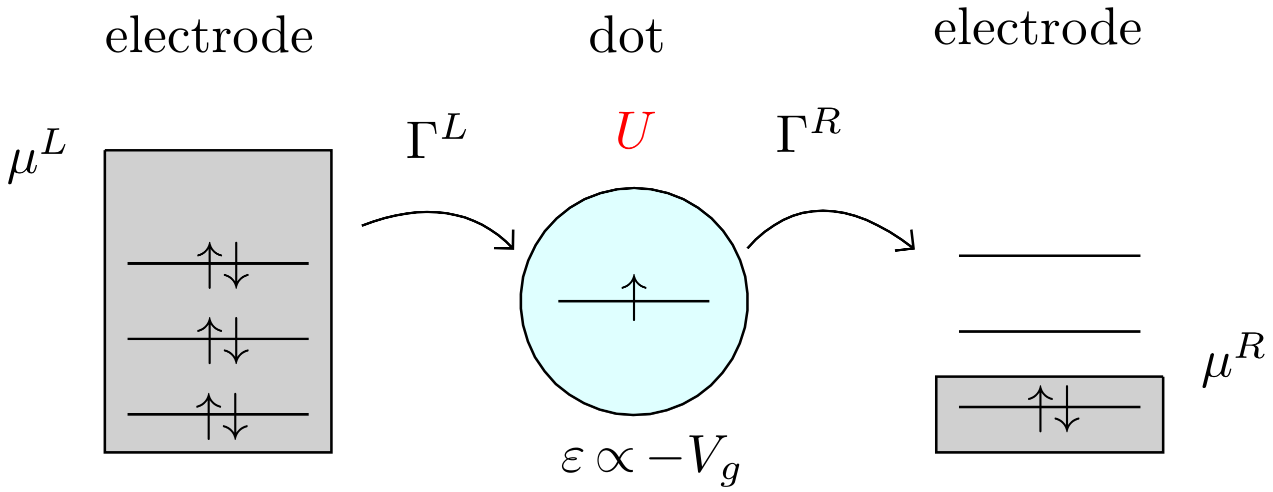
AR	**FCS**
Nonadiabatic $\hat{\rho}(t)$	"Adiabatic" $\hat{\rho}^{\chi}(t)$
Stationary $\hat{\rho}(\infty) = \hat{w}_0$	Nonstationary $\hat{\rho}^{\chi}(t)$
Landsberg phase $\hat{N}$	"Berry-Simon" phase $\hat{\rho}^{\chi}(t)$

7 Example: pumping curvature generated by interaction

Weak coupling
$W[R] \sim \Gamma \ll T$

Bias voltage $V = \mu^L - \mu^R$

Gate voltage $\varepsilon \propto -V_g$



- **Gauge away** time-dependent capacitive screening charges !

- **Tunnel rate** of electrons / holes $\propto f^{\pm}(x) = (e^{\pm x}+1)^{-1}$ through junction $r = L, R$

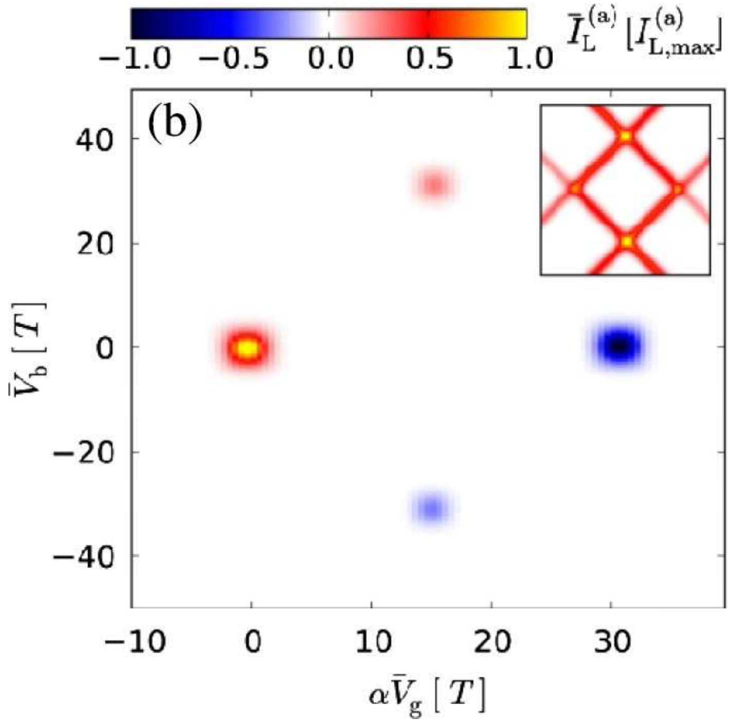
$$\begin{cases} W^r_{10} = \Gamma^r f^+([\varepsilon - \mu^r]/T), & W^r_{21} = 2\Gamma^r f^+([\varepsilon + U - \mu^r]/T) \quad \text{(tunnel in)} \\ W^r_{01} = 2\Gamma^r f^-([\varepsilon - \mu^r]/T), & W^r_{12} = \Gamma^r f^-([\varepsilon + U - \mu^r]/T) \quad \text{(tunnel out)} \end{cases}$$

- Strong Coulomb **interaction** on system

•• Explicit pumping curvature (1)

$$B = 4\,\frac{W_{10}^R + W_{12}^R}{(W_{10}+W_{12})^3}\cdot(\nabla_R W_{10})\times(\nabla_R W_{12}) + 2\left[\nabla_R \frac{W_{10}^R + W_{12}^R}{(W_{10}+W_{12})^3}\right]\cdot\left[W_{10}(\nabla_R W_{12}) - (\nabla_R W_{10})W_{12}\right]$$

with $W_{nn'} := \sum_r W_{nn'}^r$



Reckermann et. al [15]
Plücker et. al [30]

- **Geometric spectroscopy**

$$B[R] \neq 0 \quad \text{at } \textit{crossing points } R=(\varepsilon, V)$$

- **Probe new details**

$$\text{sign } B\,|_{V=U} \longleftrightarrow \begin{cases}\Gamma^L > \Gamma^R \\ \Gamma^L < \Gamma^R\end{cases} \quad \text{coupling asymmetry}$$

- **Interaction generates curvature** for fixed couplings $\{\Gamma^r\}$

$$B[R]\,\Big|_{U=0} = 0 \qquad \textit{any } R=(\varepsilon, V)$$

●● Adiabatic-response equations (Skip)

- **State** $|\rho) \sim p_n =$ occupation $n = 0, 1, 2$ charge state + rates $W_{n,n'} = W^L_{n,n'} + W^R_{n,n'}$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}|\rho) \;=\; W\,|\rho)} \quad \longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -W_{10} & W_{01} & 0 \\ W_{10} & -W_{01} - W_{21} & W_{12} \\ 0 & W_{21} & -W_{12} \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

- **Charge current** $\langle I_N \rangle$ *into* electrode r = − current *out of* system *via junction* r

$$\boxed{\langle I_N \rangle \;=\; (\mathbb{1}|W_{I_N}|\rho) \;=\; -(N|W^r|\rho)} = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -W^r_{10} & W^r_{01} & 0 \\ W^r_{10} & -W^r_{01} - W^r_{21} & W^r_{12} \\ 0 & W^r_{21} & -W^r_{12} \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

Relevant memory-kernel eigenvalue = **charge decay rate**

$$w^r_1 \;=\; -\frac{1}{2}(W^r_{10} + W^r_{12}) \qquad w_1 \;=\; \sum_r w^r_1 \;=\; -\frac{1}{2}(W_{10} + W_{12})$$

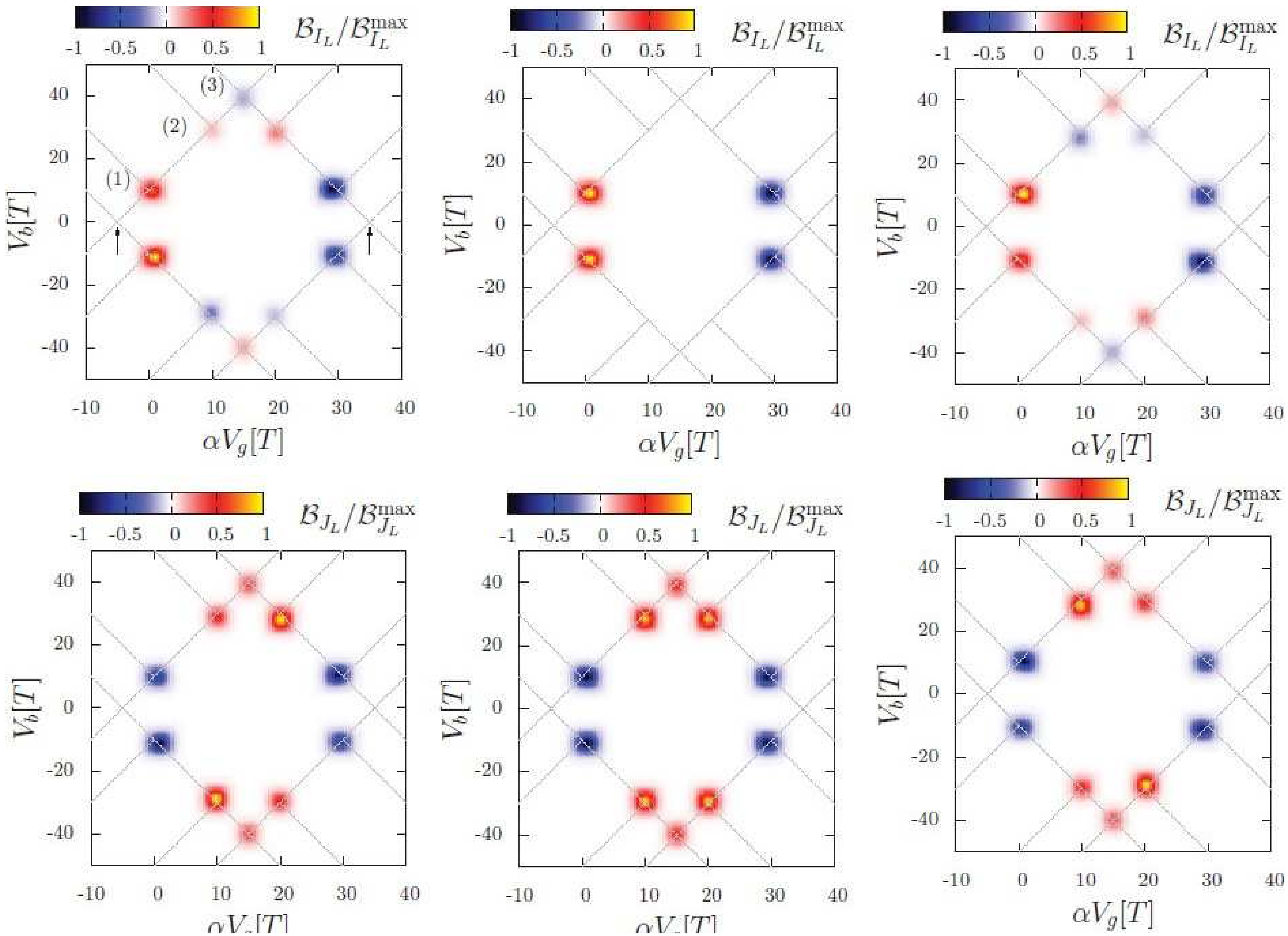
$$\boxed{A \;=\; -(N|W^r \frac{1}{W} \nabla_R|w_0) \;=\; \frac{w^r_1}{2w^3_1}\Big[W_{10}(\nabla_R W_{12}) - (\nabla_R W_{10}) W_{12} \Big]}$$

●● Explicit pumping curvature (2)

With **real external magnetic field**: Calvo, Classen et. al [7]

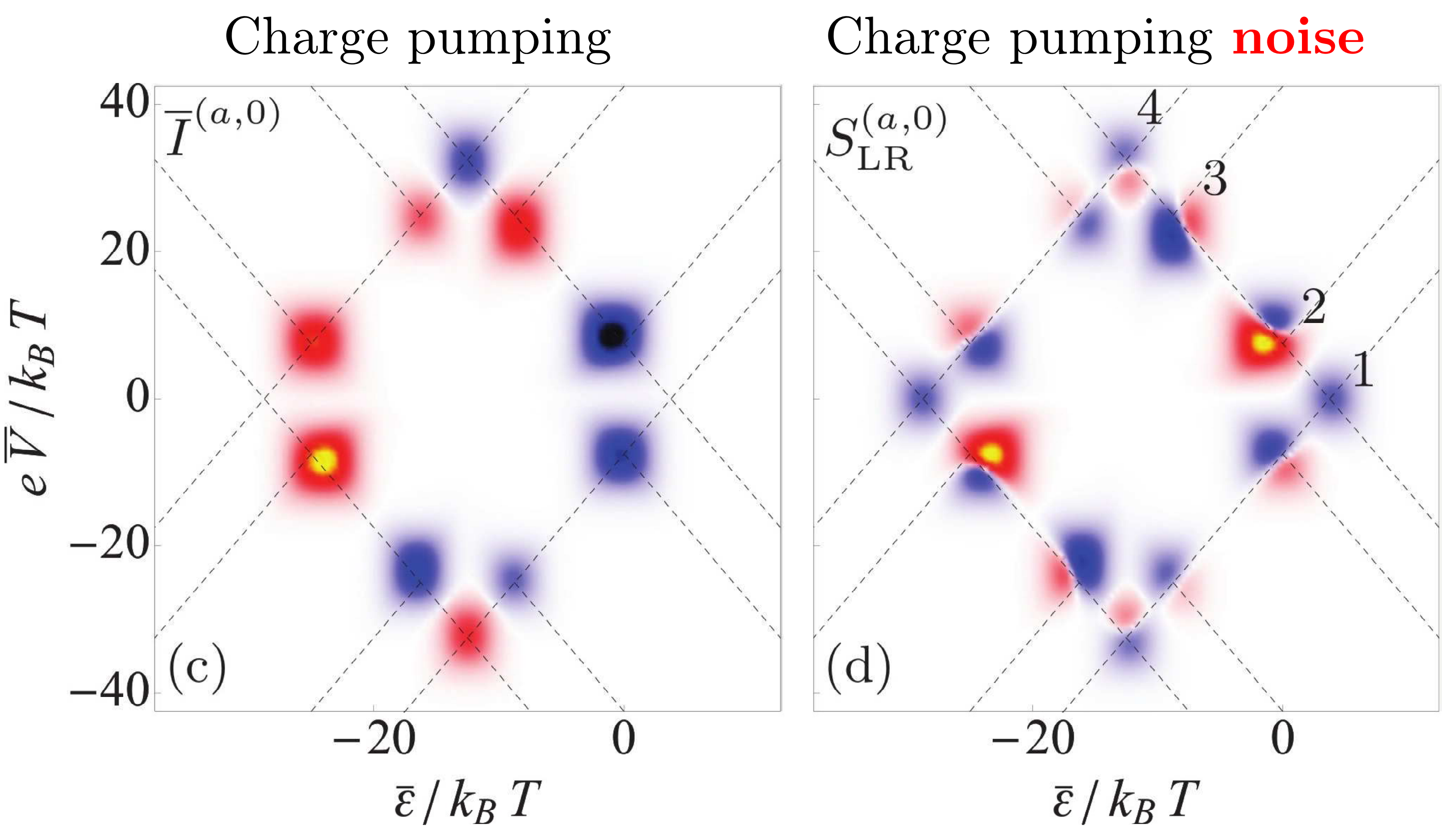
Charge pumping

Spin pumping



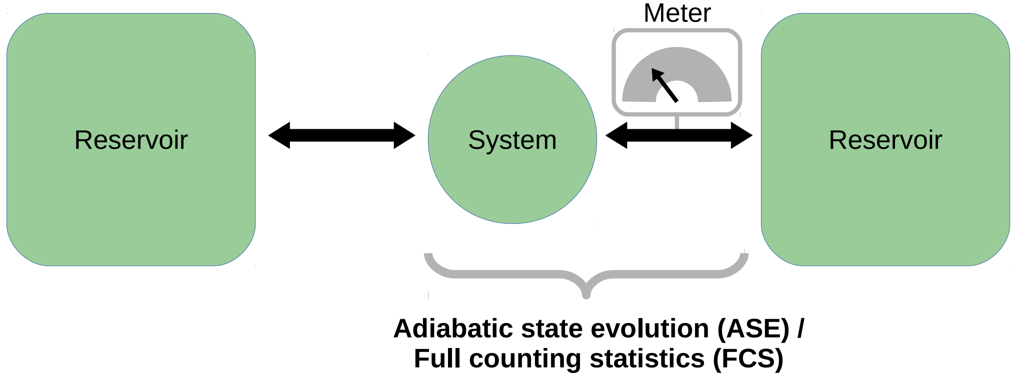
●● Explicit pumping curvature (3)

With **real external magnetic field**: Riwar et. al [18]



III Full counting statistics

Charge N measured
in reservoir r **outside** system



FCS in a nutshell:
Bagrets, Nazarov [4], Esposito, Harbola, Mukamel [8]

1. **Charge** moment $\Longleftarrow$ generating function Z^χ

$$\Delta N(t) = \langle \hat{N}(t) - \hat{N}(0) \rangle = \partial_{i\chi} Z^\chi(t) \Big|_{\chi=0}$$

2. Generating function $Z^\chi \Longleftarrow$ trace of generating operator ρ^χ

$$Z^\chi(t) = (\mathbb{1}|\rho^\chi(t)) = \mathrm{tr}\, \rho^\chi(t)$$

3. Generating operator $\rho^\chi \Longleftarrow$ "FCS master-equation"

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho^\chi) = W^\chi|\rho^\chi), \qquad |\rho^\chi(0)) = |\rho(0))$$ quantum **state** as initial condition

$$|\rho^\chi(t))\big|_{\chi=0} = |\rho(t))$$ quantum **state** included in $\chi = 0$

•• Example: cooking up FCS (Skip)

- Quantum state master equation (see → p. 41)

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}|\rho) = W\,|\rho)} \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -W_{10} & W_{01} & 0 \\ W_{10} & -W_{01}-W_{21} & W_{12} \\ 0 & W_{21} & -W_{12} \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

- Insert "**counting factor**" $e^{\pm i\chi}$ if dot charge increases / decreases *due to coupling to* R

$$W^{\pm\chi}_{n,n'} := \begin{cases} W^L_{n,n'} + e^{+i\chi} W^R_{n,n'} & n > n' \ : \ \text{enter from reservoir } R \\ W^L_{n,n'} + e^{-i\chi} W^R_{n,n'} & n < n' \ : \ \text{exit to reservoir } R \end{cases}$$

- FCS "master equation"

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}|\rho^\chi) = W^\chi\,|\rho^\chi)} \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p^\chi_0 \\ p^\chi_1 \\ p^\chi_2 \end{bmatrix} = \begin{bmatrix} -W_{10} & W^{-\chi}_{01} & 0 \\ W^{\chi}_{10} & -W_{01}-W_{21} & W^{-\chi}_{12} \\ 0 & W^{\chi}_{21} & -W_{12} \end{bmatrix} \cdot \begin{bmatrix} p^\chi_0 \\ p^\chi_1 \\ p^\chi_2 \end{bmatrix}$$

8 Sinitsyn's geometric approach to FCS

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho^\chi) = W^\chi|\rho^\chi), \qquad |\rho^\chi(0)) = |\rho(0))$$

Pumping: parametric $W^\chi(t) = W^\chi[R(t)]$ Sinitsyn, Nemenmann [26], [25], [23], [24]

$$|\rho^\chi(T)) = \textbf{?}$$

●● "Adiabatic" approximation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho^\chi) \;=\; W^\chi|\rho^\chi), \qquad |\rho^\chi(0)) \;=\; |\rho(0))$$

1. **Parametric solutions**

$$(\bar{w}_n^\chi|W^\chi \;=\; (\bar{w}_n^\chi|w_n^\chi, \qquad \longleftrightarrow \qquad W^\chi|w_n^\chi) \;=\; w_n^\chi|w_n^\chi)$$

Slowest decaying mode $n = 0$

$$0 \;<\; -\mathrm{Re}\, w_0^\chi \;<\; -\mathrm{Re}\, w_n^\chi \quad \text{for } n \geqslant 1$$

includes normalization + steady state

$$(\bar{w}_0^\chi|\bullet\Big|_{\chi=0} \;=\; (\mathbb{1}|\bullet \;=\; \mathrm{tr}\,\bullet, \qquad |w_0^\chi)\Big|_{\chi=0} \;=\; 0, \qquad |w_0^\chi)\Big|_{\chi=0} \;=\; |w_0) \quad \text{stationary state}$$

2. **Gauge freedom**

$$(\bar{w}_n{}^\chi| \;\rightarrow\; (\bar{w}_n^\chi|\cdot\frac{1}{g_n^\chi}, \qquad \longleftrightarrow \qquad |w_n^\chi) \;\rightarrow\; g_n^\chi\cdot|w_n^\chi), \qquad g_n^\chi \;\neq\; 0$$

3. **Eigenspace decoupling** for initially nonstationary state

$$c_n^\chi(0) \;=\; (\bar{w}_n^\chi|\rho(0)) \qquad |\rho^\chi(t)) \;=\; c_0^\chi(t)\cdot|w_0^\chi(t)) + \sum_{n\geqslant 1} c_n(t)\cdot|w_n^\chi(t))$$

●● "Adiabatic" generating operator – *Not* the end of story !

Slow driving:

$$\dot{R} \ll \Gamma$$

Nonzero Berry-Simon phase for $\chi \neq 0$

$$\boxed{|\rho^\chi(T)) \approx e^{\int_0^T \mathrm{d}t \left(w_0^\chi - (\bar{w}_0^\chi | \frac{\mathrm{d}}{\mathrm{d}t} | w_0^\chi) \right)} \cdot c_0^\chi(0) \cdot |w_0^\chi[R(0)]\,)} \leftarrow \text{slowest decay mode (nonstationary)}$$

$$+ \sum_{n \geqslant 1} e^{\int_0^T \mathrm{d}t \left(w_n^\chi - (\bar{w}_n^\chi | \frac{\mathrm{d}}{\mathrm{d}t} | w_n^\chi) \right)} \cdot c_n^\chi(0) \cdot |w_n^\chi[R(0)]\,) \leftarrow \text{neglect in steady state !}$$

Since we are **not** considering a quantum **state**:

(*math*) Eigenspace decoupling ⇒ **non**diabatic approximation (*physics*)

•• "Adiabatic" *steady-state* generating function

After short transient[3]

$$|\rho^\chi(T)) \approx Z^\chi \cdot |\rho^\chi(0)),$$

$\uparrow$

changed initial condition ! Nakajima et. al [14]

$$|\rho^\chi(0)) = c_0^\chi \cdot |w_0^\chi[R(0)]) \neq |\rho(0))$$

Steady state generating function:

$$\boxed{Z^\chi := e^{\int_0^T \mathrm{d}t\, w_0^\chi[R(t)] - \int_\mathcal{C} \mathrm{d}R\, A^\chi[R]}}$$

- **Eigenvalues** $\Rightarrow$ nongeometric = time-averaged FCS
- **Eigenvectors** $\Rightarrow$ geometric = pumping part with **Berry-Simon type connection**

$$A^\chi := (\bar{w}_0^\chi|\nabla_R|w_0^\chi) \quad \to \quad A^\chi_{g^\chi} = A^\chi + \frac{1}{g^\chi}\nabla_R g^\chi$$

3. $(\mathbb{1}|\rho^\chi(T)) = Z^\chi(T) \cdot (\mathbb{1}|\rho^\chi(0))$ in contrast to $(\mathbb{1}|\rho^\chi(T)) = Z^\chi(T) \cdot 1$

•• Pumped charge = Berry-Simon phase ?

Not really...

$$\Delta N = \partial_{i\chi} Z^\chi \Big|_{\chi=0}$$
$$= \partial_{i\chi} \left\{ \int_0^T \mathrm{d}t\, w_0^\chi - \int_{\mathcal{C}} \mathrm{d}R\, A_{g^\chi}^\chi \right\} \Big|_{\chi=0} \overset{\text{Stokes}}{=} \partial_{i\chi} \left\{ \int_0^T \mathrm{d}t\, w_0^\chi - \int_{\mathcal{S}} \mathrm{d}S\, B^\chi \right\} \Big|_{\chi=0}$$

- **Continuum** of contributions ?
- **Phase gradient** $\partial_{i\chi}$ at $\chi = 0$?

Pumped charge $\neq$ **single** Berry-Simon phase

•• Nonadiabatic current – included but "fragmented"

adiabatic!

$$w_0^\chi = 0 + i\chi \overset{\downarrow}{I_N}$$

$$\Downarrow$$

$$(\bar{w}_0| = (\mathbb{1}| - i\chi \underset{\uparrow}{(\Phi_N|} + \ldots \Rightarrow \boxed{\frac{\mathrm{d}}{\mathrm{d}t}|\rho^\chi) = W^\chi|\rho^\chi),} \Leftarrow |\rho^\chi) = e^{i\chi\Delta N}\Big[\overbrace{|w_0) + O(\chi)}^{|w_0^\chi)}\Big]$$

nonadiabatic — drops out

Eigenspace decoupling + collect powers of $O(1) + O(\chi)$:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\Delta N = I_N + (\Phi_N|\frac{\mathrm{d}}{\mathrm{d}t}|w_0) = I_N + \underset{\uparrow}{\dot{R}}\,(\Phi_N|\underset{\uparrow}{\nabla_R}|w_0)} = \text{AR current !}$$

nonadiabatic current !

→ p. 30 $$(\Phi_N| = (\mathbb{1}|W_{I_N} \cdot \frac{1}{W} \sim \text{current} \times \textit{finite}\ \text{relaxation time}$$

$$\boxed{\text{“Adiabatic” FCS} = \text{in fact } \textbf{non}\text{adiabatic !}}$$

•• Current memory kernel in FCS (Skip)

FCS memory kernel "knows about" **current** of AR approach:

$$W^\chi = W + i\chi W_{I_N} + \ldots$$

$\Rightarrow$ Perturbation theory in the phase χ [4]

$$(\bar{w}_0^\chi| = (\mathbb{1}| - i\chi\,(\underset{\substack{\uparrow \\ \text{nonadiabatic}}}{\Phi_N}| + \ldots, \qquad w_0^\chi = 0 + i\chi\,\underset{\substack{\uparrow \\ \text{adiabatic !}}}{I_N}, \qquad |w_0^\chi) = |w_0) + \underset{\substack{\uparrow \\ \text{drops out}}}{O(\chi)}$$

4. Adiabatic current $I_N = (\mathbb{1}|W_{I_N}|w_0)$ = matrix element perturbation in unperturbed basis

●● Example $W^{\chi} = W + i\chi W_{I_N} + \ldots$ (Skip)

FCS "master equation": $W^{\pm}_{n,n'} = W_{n,n'} \pm i\chi W^{R}_{n,n'} + \ldots$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}|\rho^{\chi}) \;=\; W^{\chi}|\rho^{\chi})} \quad \longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -W_{10} & W_{01}^{-\chi} & 0 \\ W_{10}{}^{\chi} & -W_{01} - W_{21} & W_{12}^{-\chi} \\ 0 & W_{21}{}^{\chi} & -W_{12} \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

- O(1): **state**-evolution memory kernel

$$W \;=\; \begin{bmatrix} -W_{10} & W_{01} & 0 \\ W_{10} & -W_{01} - W_{21} & W_{12} \\ 0 & W_{21} & -W_{12} \end{bmatrix}$$

- O(χ): **current** memory kernel *up to a irrelevant term* Plücker et al. [30] with $(\mathbb{1}|\Theta|\rho) = 0$ for any ρ)

$$W_{I_N} + \Theta \;=\; \begin{bmatrix} 0 & -W^{R}_{01} & 0 \\ +W^{R}_{10} & 0 & -W^{R}_{12} \\ 0 & +W^{R}_{21} & 0 \end{bmatrix} \quad \leftarrow \text{exit to lead } R$$

$\uparrow$

enter from lead R

●● Guiding questions

1. **Fiber bundle?** Total space

$$(\ R\ ,\ g^\chi\ )\ =\ \left(\begin{array}{c}\text{driving}\\ \text{parameter}\end{array},\ \begin{array}{c}\textbf{slowest-mode}\\ |w_0^\chi)\ \text{normalization}\end{array}\right)$$

2. **Connection?** **Broken** physical solution curve

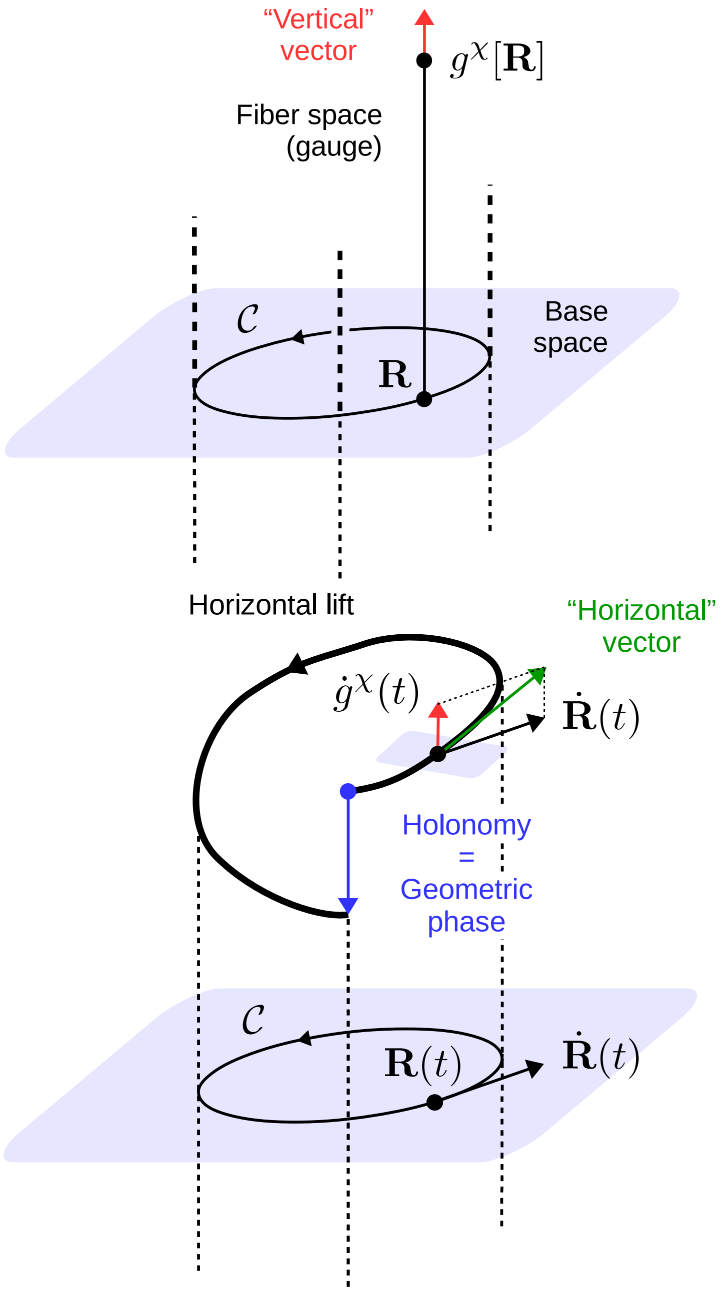
$$g^\chi(T)\ =\ e^{-\int_0^T \mathrm{d}t\, \dot{R}(t)\cdot A^\chi[R(t)]}\cdot g^\chi(0)\ \propto\ \boxed{Z^\chi(T)}$$

⇒ **Unavoidable effects** on the **entire transport process** due to cyclic driving

"Horizontal lift" = **Try to** "gauge away" A^χ:

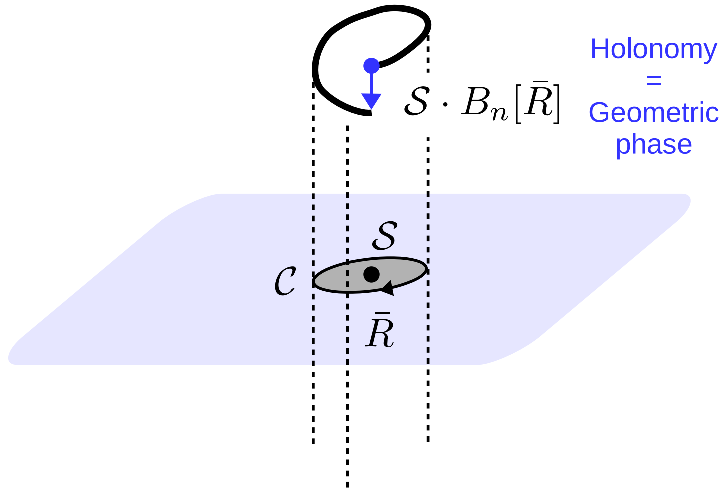
$$\dot{R}\cdot A^\chi_{g^\chi}\ =\ \dot{R}\cdot A^\chi + \frac{1}{g^\chi}\cdot\dot{g}^\chi\ =\ \boxed{\frac{\mathrm{d}}{\mathrm{d}t}(\ln Z^\chi + \dots)\ \stackrel{?}{=}\ 0}$$

= "recalibrate all meters" of **entire transport process** to register **zero pumping**: average ΔN, noise ΔN^2, ...



•• FCS curvature (Skip)

2. **Curvature?**
   = **breakage** of "horizontal lift" curves



$$B^\chi \;=\; \nabla_R \times A^\chi \;=\; (\nabla_R \bar{w}_0^\chi| \times |\nabla_R w_0^\chi)$$

For infinitesimal curve $\mathcal{C}$ around $\bar{R}$ enclosing area $\mathcal{S} \to 0$

$$Z^\chi \;=\; e^{\int_0^T \mathrm{d}t\, w_0^\chi - \int_\mathcal{C} \mathrm{d}R\, A^\chi} \;\overset{\text{Stokes}}{\approx}\; Z^\chi[\bar{R}] \cdot \Big(1 - \mathcal{S} \cdot B^\chi(\bar{R})\Big)$$

↑ generating function for average parameter $\bar{R}$

9 Questions

Advantages:

1. **Entire transport process** described

$$\left\langle\, \mathcal{T}\Big(\hat{N}(t)-\hat{N}(0)\Big)^k \right\rangle \;=\; (\partial_{i\chi})^k\, Z^\chi\Big|_{\chi=0}$$

2. **Formal** analogy to **closed systems**

$$|\rho^\chi(T)) \;\approx\; e^{\int_0^T \mathrm{d}t\, w_0^\chi - \int_{\mathcal{C}} \mathrm{d}R\, A^\chi} \cdot |\rho^\chi(0))$$

.... but **physical** questions remain ...

Adiabatic mixed-**state** evolution of a **physical system** ?

IV Adiabatic state evolution

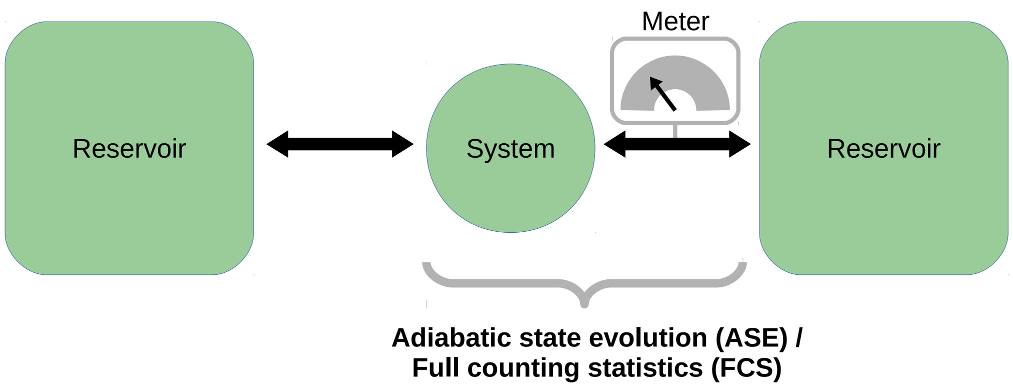
$$\text{Part IV} = \text{Part I} + \text{meter} = \text{Part II (AR)}$$
$$= \text{Part III (FCS)}$$

Two steps:

1. Ideal charge **meter**
   Levitov [12], Schaller, Kiesslich, Brandes [21]

   $\hat{N}$:= charge indicated by **meter** (not on reservoir as before)

2. Adiabatic **state** evolution
   Sarandy, Lidar [19], [20]

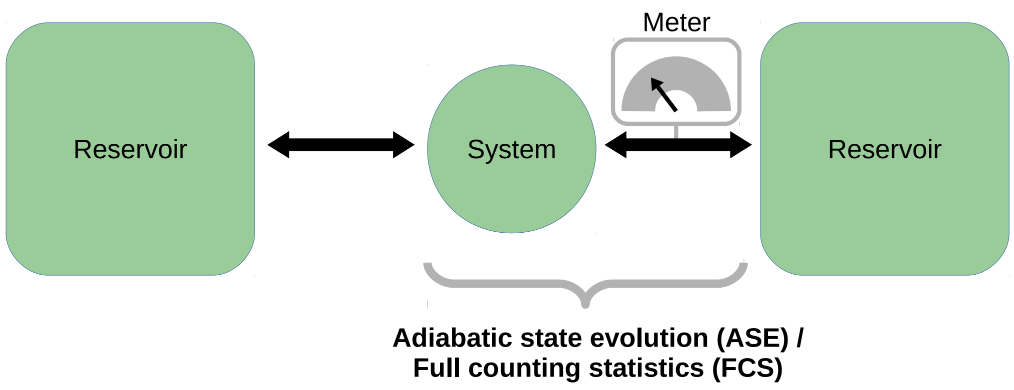


●● Step 1: Coupling system + meter – Hamiltonian

Model (AR, FCS)

$$\hat{H}^{\text{tot}} = \hat{H} + \hat{H}^{\text{R}} + \sum_n \hat{V}^{-n}$$

$\hat{V}^n =$ part of coupling V transferring n charges to **specific** reservoir



Extended model: ideal meter with meter states $\{|n\rangle\}$ and no dynamics

$$\hat{H}^{\text{tot}\prime} = \hat{H} + \hat{H}^{\text{R}} + 0 + \sum_n \hat{V}^{-n} \otimes e^{-i\hat{\chi}n}$$

- Charge translation on meter $\Leftarrow$ **phase operator** $\hat{\chi}$

$$e^{-i\hat{\chi}n}|m\rangle = |m+n\rangle$$

- Initial meter state = pure state $|0\rangle$

●● Step 1: Coupling system + meter – memory kernel

System + meter mixed state ρ'

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho') = W'|\rho')$$
$$W' = \sum_n W^{-n} \otimes e^{-iL_{\hat{\chi}}n} = \int_{-\pi}^{\pi} \frac{\mathrm{d}\chi}{2\pi} W^{\chi} \otimes |\chi)(\chi|$$

- Charge-translation $\Leftarrow$ **phase superoperator** $L_{\hat{\chi}} := \hat{\chi} \bullet - \bullet \hat{\chi}$

$$e^{-iL_{\hat{\chi}}n}|m\rangle\langle m| = e^{-i\hat{\chi}n}|m\rangle\langle m|e^{i\hat{\chi}n} = |m+n\rangle\langle m+n|$$

- $\Rightarrow$ Fourier transform to **phase eigenmode**

$$|\chi) = \sum_{n=-\infty}^{\infty} e^{i\chi n}|n\rangle\langle n|, \qquad L_{\hat{\chi}}|\chi) = \chi|\chi),$$

"plane waves in charge space"

$$|k\rangle \propto \int \mathrm{d}x\, e^{ikx}|x\rangle$$

- $\Rightarrow$ Fourier transform to **FCS memory kernel** !

$$W^{\chi} = \sum_n e^{in\chi} W^n$$

●● Step 1: Coupling system + meter – mixed state

System + meter mixed state ρ'

$$|\rho') \;=\; \sum_{n=-\infty}^{\infty} |\rho^{-n}) \otimes |n\rangle\langle n| \;=\; \int_{-\pi}^{\pi} \frac{d\chi}{2\pi}\, |\rho^{\chi}) \otimes |\chi) \quad \uparrow$$

! FCS $\quad |\rho^{\chi}) \;=\; \sum_{n} e^{in\chi}\, |\rho^{n})$

"wave function of the meter"

$$|\psi\rangle \propto \int dx\, \psi(x)\, |x\rangle = \int dk\, \psi_k\, |k\rangle$$

●● Step 2: Adiabatic state evolution *system + meter*

Adiabatic system + meter mixed state ρ' Sarandy, Lidar [19], [20]

$$|\rho'(T)) \;=\; \int_{-\pi}^{\pi} \frac{\mathrm{d}\chi}{2\pi}\, |\rho^{\chi}(T)) \otimes |\chi)$$

$\uparrow$

! "Adiabatic" FCS $\quad |\rho^{\chi}(T)) \;=\; Z^{\chi} \cdot |\rho^{\chi}(0))$

$\uparrow$

$$Z^{\chi} \;=\; e^{\int_0^T \mathrm{d}t\, w_0^{\chi} - \int_C \mathrm{d}R\, A^{\chi}}$$

Reminder from Part I → p. 22:

Berry-Simon phase in parametrically **nonstationary** part system + meter state !

●● Step 2: Adiabatic state evolution *system + meter*

Adiabatic system + meter mixed state ρ' Sarandy, Lidar [19], [20]

$$|\rho'(T)) \;=\; \int_{-\pi}^{\pi} \frac{\mathrm{d}\chi}{2\pi}\, |\rho^{\chi}(T)) \otimes |\chi)$$

$\uparrow$

! "Adiabatic" FCS: $|\rho^{\chi}(T)) \;=\; Z^{\chi} \cdot |\rho^{\chi}(0))$

$\uparrow$

$$Z^{\chi} \;=\; e^{\int_0^T \mathrm{d}t\, w_0^{\chi} - \int_C \mathrm{d}R\, A^{\chi}}$$

- $\chi \neq 0$: **Charge meter keeps running**

  FCS $\quad |\rho^{\chi}(T)) \sim$ **nonstationary** $\Rightarrow$ Berry-Simon phase ($\rightarrow$ p. 22)

- $\chi = 0$: **Stop the charge meter**

  AR $\quad |\rho^{\chi}(T))\big|_{\chi=0} \;=\; |\rho(T)) \;=\; 1 \cdot |w_0[R(0)]\,)$ **stationary** $\Rightarrow$ **no** phase ($\rightarrow$ p. 21)

  $\uparrow$

  probability conservation

•• FCS: Pumped charge = Berry-Simon phase ?

$$\boxed{\Delta N \;=\; \underset{\mathrm{M}}{\mathrm{tr}}\,\mathrm{tr}\,\underset{\uparrow}{\hat{N}}\Big(\hat{\rho}'(T)-\hat{\rho}'(0)\Big)} = \underset{\uparrow}{\partial_{i\chi}}\left\{\int_0^T \mathrm{d}t\, w_0^{\chi} - \int_{\mathcal{C}} \mathrm{d}R\, A^{\chi}_{g^{\chi}}\right\}\Bigg|_{\chi=0}$$

FCS result !

$$\hat{N} := \sum_n n\,|n\rangle\langle n| \qquad \text{meter charge}$$

- **Continuum** = **unavoidable**: one Berry-Simon phase for each meter "momentum" (χ) !

$$|\rho'(T)) \;=\; \int_{-\pi}^{\pi}\frac{\mathrm{d}\chi}{2\pi}\,|\rho^{\chi}(T))\otimes|\overset{\downarrow}{\chi})$$

- **Phase gradient** $\partial_{i\chi}$ = **unavoidable**: meter charge operator in phase representation !

•• AR: Landsberg's phase = accumulated by *observable* ?

$$|\rho'(T)) \;=\; \int_{-\pi}^{\pi} \frac{\mathrm{d}\chi}{2\pi}\,\frac{1}{g^{\chi}}|\rho^{\chi}(T)) \otimes \boxed{g^{\chi}\,|\chi)}$$

Ideal meter

- Gauge transformations include

$$g^{\chi} \;=\; e^{i\chi\, g}$$

- Phase eigenmode ∼ generated by conjugate observable = **charge** !

$$|\chi) \;=\; \sum_n e^{i\chi\, n}|n\rangle\langle n| \;=\; e^{i\,\hat{N}\chi}$$

"plane wave in charge space"

Gauge **meter**-part of **state** = recalibration **charge observable** cf. → p. 31

$$\text{(FCS / ASE)} \quad \boxed{g^{\chi}\,|\chi) \;=\; e^{i\,(\hat{N}+g)\chi}} \quad \text{(AR)}$$

AR	**FCS**	**ASE + meter**
Nonadiabatic $\rho(t)$	"Adiabatic" $\rho^{\chi}(t)$	Adiabatic state $\rho'(t)$
Stationary $\rho(\infty) = w_0$	Nonstationary $\rho^{\chi}(t)$	Nonstationary $\rho'(t)$
Landsberg phase $\hat{N}$	"Berry-Simon" phase $\rho^{\chi}(t)$	Berry-Simon phases in $\rho'(t)$

V Summary

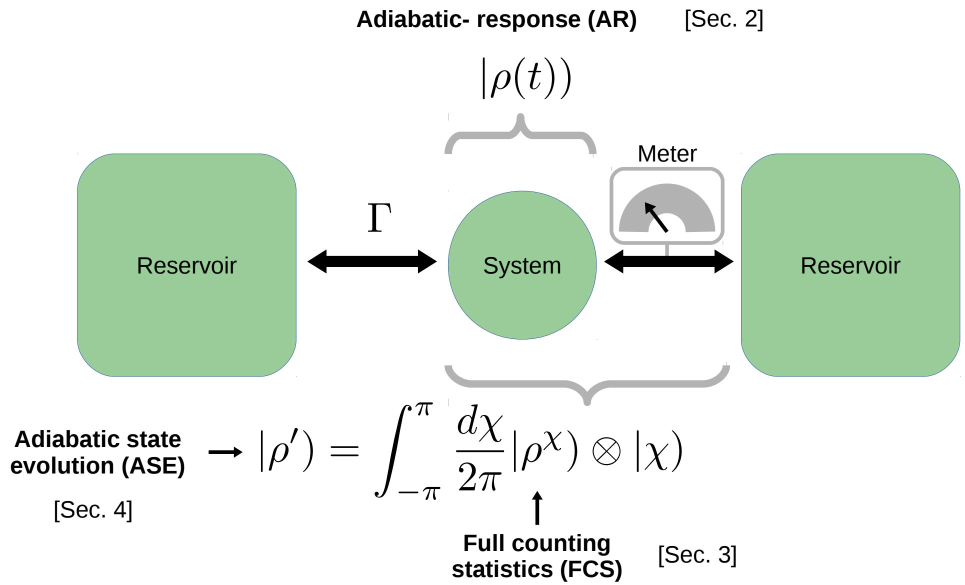
Geometric pumping in open systems

1. **Not** "a" Berry-Simon phase:
   - Landsberg phase
   - *Phase-gradient of a continuum of* Berry-Simon phases

2. **Non**adiabatic / "lag":
   Time-window for charge response

3. Steady state:
   - System nearly **stationary**
   - Meter **nonstationary**

4. Gauge freedom = **meter recalibration**
   - Meter outside: $\hat{N} \to \hat{N} + g$
   - Meter inside: $|\chi) \to e^{ig\chi}|\chi)$



Measurement essential to **physically** understand **geometric** effects

*"You can **time-dependently** mess around all you like with the scale of an $\hat{X}$-**meter**, as long as you **periodically return it** to **measure** the correct pumped value of $\hat{X}$."*

$\hat{X}$ = charge, spin, energy / heat, ...

Geometry / physics = independent of coordinates

- **Geometric / topological pumping**
  $\Rightarrow$ Connections derived from driven **open-system dynamics + measurements**
  - Adiabatic-response:
    Landsberg connection
  - Full counting statistics / Adiabatic state evolution + meter
    Berry-Simon connection of **Sinitsyn** / **Sarandy-Lidar**

- **Topological classification mixed states**
  $\Rightarrow$ **Uhlmann** connection = derived from **mixed-state distance** measure (QI)
  Martin-Delgado et. al [29], Arovas et. al [9], Budich, Diehl, [6]

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